A Few Good Examples why to Refute Classical Logic in Quantum Mechanics

Lovre Grisogono
Mateo Pauliŝić

University of Zagreb
Faculty of Science
Department of Physics

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Classical Logic

- Classical logic is a type of formal logic which is most intensively studied and most widely used on all levels of education.
- Classical logic is defined by a set of properties.
- Classical logic generally includes propositional logic and first-order logic.

Definition of CL

We define classical logic with the following set of properties:

- Binarity:
  Set of truth values - $\mathcal{V}(P) \in \{0, 1\}$
- Commutativity:
  $P \wedge Q \equiv Q \wedge P$, $P \vee Q \equiv Q \vee P$
- Distributivity:
  $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
  $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
- Principle of excluded middle:
  $\mathcal{V}(P \vee \neg P) = 1$
- Principle of non-contradiction:
  $\mathcal{V}(\neg(P \wedge \neg P)) = 1$, alternatively $\mathcal{V}(P \wedge \neg P) = 0$

Quantum Mechanics

- Branch of physics
- Atomar and subatomar scale
- Commutative and non-commutative operators
- Uncertainty principle

But the Question Is?

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Should we refute classical logic for QM?

Definitions

- Quantum mechanical proposition is a projection operator (generally observable) P
  - experiment outcome prediction
- $Px = x = 1 \cdot x$ for $x \in L_P$
  $Py = 0 = 0 \cdot y$ for $y \in L_P^{\perp}$
- $P \wedge Q$ - operator projecting on $L_{P \wedge Q} = L_P \cap L_Q$
- $P \vee Q$ - operator projecting on $\overline{L_P + L_Q}$
- $\mathbf{\neg P = 1 - P}$
- $\mathbf{P \wedge Q = \lim_{n \to \infty} (PQ)^n}$
- $\mathbf{P \vee Q = \neg[\neg P \wedge \neg Q]}$

Examples

- Example 1: $S_z = (\frac{\hbar}{2})P_1 - (\frac{\hbar}{2})P_2$

$P_1 = |\uparrow\rangle\langle\uparrow| \quad P_2 = |\downarrow\rangle\langle\downarrow|$

Matrix notation: $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

For S_z:

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Using previous definitions:

- $$\mathbf{P_1 \wedge P_2 = \lim_{n\to+\infty}(P_1P_2)^n}$$
$$= \lim_{n\to+\infty}\left[\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\cdot\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right]^n$$
$$= \lim_{n\to+\infty}\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}^n = \mathbf{0}$$

- $$\mathbf{P_1 \vee P_2 = 1 - [(1 - P_1) \wedge (1 - P_2)]}$$
$$= 1 - [P_2 \wedge P_1]$$
$$= \mathbf{1}$$

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Calculations were simplified due to $[P_1, P_2] = 0$

Example of non-commuting observables:

- $S_x = \left(\frac{\hbar}{2}\right) Q_1 - \left(\frac{\hbar}{2}\right) Q_2$, it is easily seen that:

  $$Q_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad Q_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

  are projectors projecting on subspaces $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
- Operators Q_1 and Q_2 do not commute with P_1 and P_2
- $$\mathbf{P_1} \wedge \mathbf{Q_1} = \lim_{n \to +\infty} (P_1 Q_1)^n = \lim_{n \to +\infty} \left(\tfrac{1}{2}\right)^n \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{0}$$
- $$\begin{aligned} \mathbf{P_1} \vee \mathbf{Q_1} &= 1 - ((1 - P_1) \wedge (1 - Q_1)) \\ &= 1 - (P_2 \wedge Q_2) \\ &= 1 - 0 = \mathbf{1} \end{aligned}$$

Non-commutativity $\neq$ zero conjunction

Example: particle with spin 1

- $B = \frac{1}{2\hbar^2}[S^2 - S_z^2] + \frac{1}{2\hbar^2}S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
- $D = \frac{1}{\hbar^2}S_z^2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$, $[B, D] \neq 0$ but $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is a common eigenvector.
- $D \wedge B = \lim_{n \to +\infty} \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Failure of distributivity:

- Easily calculated as before:
  $P_1 \vee P_2 = 1 \quad Q_1 \wedge P_1 = 0 \quad Q_1 \wedge P_2 = 0$

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Comment: Modular law holds:

- $C \to A$
- $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C) \equiv (A \wedge B) \vee C$

Failure of continuity - example using polarization

- let $P_\epsilon = \begin{pmatrix} \cos^2\epsilon & \cos\epsilon\sin\epsilon \\ \cos\epsilon\sin\epsilon & \sin^2\epsilon \end{pmatrix}$

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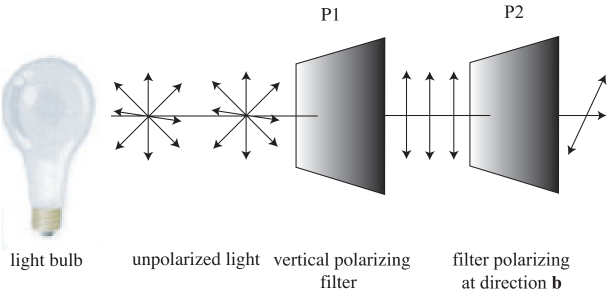
- for $\epsilon \neq 0$, $P_\epsilon \wedge P_1 = 0$
- $\lim_{\epsilon\to 0}(P_\epsilon \wedge P_1) = 0$
- $(\lim_{\epsilon\to 0} P_\epsilon) \wedge P_1 = P_1 \wedge P_1 = P_1$

Polarization

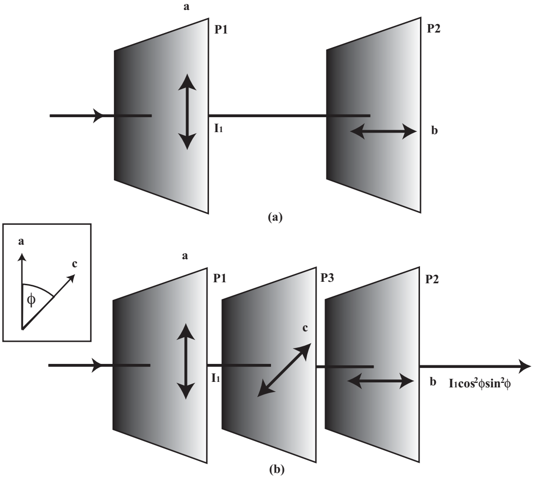
Figure: Polarization example



- Polarizing filter = Mechanical analogue to the projection operator

Polarization

Figure: Polarization example



Polarization

- H - horizontal filter, V - vertical filter, D - diagonal filter
- $H^n = H,\ V^n = V,\ HV = 0$

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- $VHD \neq VDH$
- $H \wedge V = \lim_{n \to \infty} (HV)^n = 0$
- $V \wedge D = \lim_{n \to \infty} (VD)^n = 0$
- $$D \vee V = \neg[\neg D \wedge \neg V] \\ = \neg[D_{anti} \wedge H] = \neg 0 = 1$$

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Failure of distributivity

- $H \wedge (D \vee V) \equiv H \wedge 1 \equiv H$

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- $H \wedge (D \vee V) \equiv H \wedge 1 \equiv H$
- $(H \wedge D) \vee (H \wedge V) \equiv 0 \vee 0 \equiv 0$

Non-continuity of conjunction

- D_ϵ - polarizing filter inclined at ϵ to the horizontal
  $\lim_{\epsilon \to 0} D_\epsilon = H$

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- $D_\epsilon \wedge H = 0$
- $\lim_{\epsilon \to 0} (D_\epsilon \wedge H) = 0$
- $(\lim_{\epsilon \to 0} D_\epsilon) \wedge H = H$

Two slit experiment - failure of distributivity

- A_1 - electron passes through slit 1
  A_2 - electron passes through slit 2
- $P(A_1, R)$ - probablity that electron hits region R, after A_1
  $P(A_2, R)$ - probablity that electron hits region R, after A_2
- Classically, one would expect
  $P(R) = \frac{1}{2}P(A_1, R) + \frac{1}{2}P(A_1, R)$

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- Classically, one would expect
  $P(R) = \frac{1}{2}P(A_1, R) + \frac{1}{2}P(A_1, R)$
- *Quantum formula* $P(R) = \frac{1}{2}P(A_1, R) + \frac{1}{2}P(A_1, R) + \sqrt{P(A_1, R)P(A_2, R)} \cos \delta$

- $$P(A_1 \vee A_2, R) = \frac{P((A_1 \vee A_2) \cdot R)}{P(A_1 \vee A_2)} = \frac{P((A_1 \cdot R) \vee (A_2 \vee R))}{P(A_1 \vee A_2)} = \frac{P(A_1 \cdot R)}{P(A_1 \vee A_2)} + \frac{P(A_2 \cdot R)}{P(A_1 \vee A_2)}$$

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- $P(A_1) = P(A_2)$
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- $$\frac{P(A_1 \cdot R)}{P(A_1 \vee A_2)} = \frac{P(A_1 \cdot R)}{2P(A_1)} = \frac{1}{2}P(A_1, R)\ ;$$
  $$\frac{P(A_2 \cdot R)}{P(A_1 \vee A_2)} = \frac{P(A_2 \cdot R)}{2P(A_2)} = \tfrac{1}{2}P(A_2, R)$$
- $P(R) = \frac{1}{2}P(A_1, R) + \frac{1}{2}P(A_1, R)$

Conclusion

The question was:

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And the anwser is:

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And the anwser is:

We should.

Literature

- 'Quantum mechanics' [Auletta, G., Fortunato, M., Parisi, G. Quantum mechanics, Cambridge University Press, 2009]
- 'Quantum logic' [Adler, C. G., Wirth, J. F. Quantum Logic. In American Journal of Physics 51(5): 1983, pp. 412-417.]
- 'Is logic empirical' [Putnam, H. Is Logic Empirical? In Boston Studies in the Philosophy of Science 5, R. S. Cohen, M. W. Wartofsky, ur. Dodrecht: D. Reidel, 1968, pp. 174-197.]