**Zadatak 6.4** Nađite jednostavni verižni (neprekidni) razlomak od:

**1.** $\frac{2+\sqrt{5}}{3} = \left[\overline{1,2,2,2,1,12}\right]$ .

- $$\alpha = \frac{2+\sqrt{5}}{3} \Longrightarrow s_0 = 2,\ t_0 = 3,\ 3 \nmid \left(5-2^2\right) \Longrightarrow$$

$$\alpha = \frac{6+\sqrt{45}}{9} \Longrightarrow s_0 = \mathbf{6},\ t_0 = \mathbf{9},\ 9 \,\Big|\, \left(45-6^2\right)$$

$$a_0 = \left\lfloor \alpha_0 \right\rfloor = \left\lfloor \frac{s_0+\sqrt{45}}{t_0} \right\rfloor = \left\lfloor \frac{6+\sqrt{45}}{9} \right\rfloor \overset{\alpha_0 \simeq 1.4}{=} 1,$$

$$s_1 = a_0 t_0 - s_0 = 1\cdot 9 - 6 = \mathbf{3},\ t_1 = \frac{45-s_1^2}{t_0} = \frac{45-3^2}{9} = \mathbf{4}$$

- $$a_1 = \left\lfloor \alpha_1 \right\rfloor = \left\lfloor \frac{s_1+\sqrt{45}}{t_1} \right\rfloor = \left\lfloor \frac{3+\sqrt{45}}{4} \right\rfloor \overset{\alpha_1 \simeq 2.4}{=} 2,$$

$$s_2 = a_1 t_1 - s_1 = 2\cdot 4 - 3 = \mathbf{5},\ t_2 = \frac{45-s_2^2}{t_1} = \frac{45-5^2}{4} = \mathbf{5}$$

- $$a_2= \left\lfloor \alpha_2 \right\rfloor = \left\lfloor \frac{s_2+\sqrt{45}}{t_2} \right\rfloor = \left\lfloor \frac{5+\sqrt{45}}{5} \right\rfloor \overset{\alpha_2 \simeq 2.3}{=} 2,$$
$$s_3= a_2 t_2 - s_2 = 2 \cdot 5 - 5 = \mathbf{5},\ \ t_3 = \frac{45 - s_3^2}{t_2} = \frac{45-5^2}{5} = \mathbf{4}$$

- $$a_3= \left\lfloor \alpha_3 \right\rfloor = \left\lfloor \frac{s_3+\sqrt{45}}{t_3} \right\rfloor = \left\lfloor \frac{5+\sqrt{45}}{4} \right\rfloor \overset{\alpha_3 \simeq 2.9}{=} 2,$$
$$s_4= a_3 t_3 - s_3 = 2 \cdot 4 - 5 = \mathbf{3},\ \ t_4 = \frac{45 - s_4^2}{t_3} = \frac{45-3^2}{4} = \mathbf{9}$$

- $$a_4= \left\lfloor \alpha_4 \right\rfloor = \left\lfloor \frac{s_4+\sqrt{45}}{t_4} \right\rfloor = \left\lfloor \frac{3+\sqrt{45}}{9} \right\rfloor \overset{\alpha_4 \simeq 1.07}{=} 1,$$
$$s_5= a_4 t_4 - s_4 = 1 \cdot 9 - 3 = \mathbf{6},\ \ t_5 = \frac{45 - s_5^2}{t_4} = \frac{45-6^2}{9} = \mathbf{1}$$

- $$a_5= \left\lfloor \alpha_4 \right\rfloor = \left\lfloor \frac{s_5+\sqrt{45}}{t_5} \right\rfloor = \left\lfloor \frac{6+\sqrt{45}}{1} \right\rfloor \overset{\alpha_5 \simeq 12.7}{=} 12,$$
$$s_6= a_5 t_5 - s_5 = 12 \cdot 1 - 6 = \mathbf{6},\ \ t_6 = \frac{45 - s_6^2}{t_5} = \frac{45-6^2}{1} = \mathbf{9}$$

- $$(s_0, t_0) = (s_6, t_6) = (\mathbf{6}, \mathbf{9}) \implies j = 0,\ k = 6$$

$$\overset{r=6-\text{period}}{\Longrightarrow} \quad \alpha = [a_0, \ldots, a_{j-1}, \overline{a_j, \ldots, a_{k-1}}] =$$

$$= [\overline{a_0, \ldots, a_{6-1}}] = \left[\overline{\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{12}}\right].$$

**2.** $\sqrt{\frac{2c+1}{2c}} = \left[1, \overline{4c, 2}\right], \, c \in \mathbb{N};$

- $$\alpha \; = \; \sqrt{\frac{2c+1}{2c}} = \frac{0 + \sqrt{2c\,(2c+1)}}{2c} \Longrightarrow$$

$$s_0 \; = \; \mathbf{0}, \;\; t_0 = \mathbf{2c}, \;\; 2c \,\Big|\, \Big(2c\,(2c+1) - 0^2\Big)$$

- $$a_0 = \Big\lfloor \alpha_0 \Big\rfloor = \Bigg\lfloor \frac{s_0 + \sqrt{2c\,(2c+1)}}{t_0} \Bigg\rfloor = \Bigg\lfloor \frac{0 + \sqrt{2c\,(2c+1)}}{2c} \Bigg\rfloor$$

$$\alpha_0 = \frac{0 + \sqrt{2c\,(2c+1)}}{2c} = \sqrt{\frac{2c+1}{2c}} = \sqrt{1 + \frac{1}{2c}}$$

$$1 < \; \sqrt{1 + \frac{1}{2c}} < 2 \; \Longrightarrow \; a_0 = \Big\lfloor \alpha_0 \Big\rfloor = 1$$

$$s_1 = a_0 t_0 - s_0 = 1 \cdot 2c - 0 = \mathbf{2c},$$

$$t_1 = \frac{2c\,(2c+1) - s_1^2}{t_0} = \frac{2c\,(2c+1) - (2c)^2}{2c} = \mathbf{1}$$

- $$a_1 = \left\lfloor \alpha_1 \right\rfloor = \left\lfloor \frac{s_1 + \sqrt{2c\,(2c+1)}}{t_1} \right\rfloor = \left\lfloor \frac{2c + \sqrt{2c\,(2c+1)}}{1} \right\rfloor$$

$$\alpha_1 = 2c + \sqrt{2c\,(2c+1)}$$

$$2c = \sqrt{(2c)^2} < \sqrt{2c\,(2c+1)} < \sqrt{(2c+1)^2} = 2c+1 \implies$$

$$4c = 2c + 2c < 2c + \sqrt{2c\,(2c+1)} < 2c + 2c + 1 = 4c + 1 \implies$$

$$a_1 = \left\lfloor \alpha_1 \right\rfloor = 4c$$

$$s_2 = a_1 t_1 - s_1 = 4c \cdot 1 - 2c = \mathbf{2c},$$

$$t_2 = \frac{2c\,(2c+1) - s_2^2}{t_1} = \frac{2c\,(2c+1) - (2c)^2}{1} = \mathbf{2c}$$

- $$a_2 = \left\lfloor \alpha_2 \right\rfloor = \left\lfloor \frac{s_2 + \sqrt{2c\,(2c+1)}}{t_2} \right\rfloor = \left\lfloor \frac{2c + \sqrt{2c\,(2c+1)}}{2c} \right\rfloor$$

$$\alpha_2 = \frac{2c + \sqrt{2c\,(2c+1)}}{2c} = 1 + \sqrt{\frac{2c+1}{2c}} = 1 + \sqrt{1 + \frac{1}{2c}}$$

$$2 < 1 + \sqrt{1 + \frac{1}{2c}} < 3 \implies$$

$$a_2 = \lfloor \alpha_2 \rfloor = \left\lfloor 1 + \sqrt{1 + \frac{1}{2c}} \right\rfloor = 2$$

$$s_3 = a_2 t_2 - s_2 = 2 \cdot 2c - 2c = \mathbf{2c},$$

$$t_2 = \frac{2c\,(2c+1) - s_3^2}{t_2} = \frac{2c\,(2c+1) - (2c)^2}{2c} = \mathbf{1}$$

- $$(s_1, t_1) = (s_3, t_3) = (\mathbf{2c}, \mathbf{1}) \implies j = 1,\ k = 3$$

$$\overset{r=2-\text{period}}{\implies} \quad \alpha = [a_0, \ldots, a_{j-1}, \overline{a_j, \ldots, a_{k-1}}] =$$

$$= [a_0, \overline{a_1, a_2}] = \left[\mathbf{1}, \overline{\mathbf{4c}, \mathbf{2}}\right].$$

- $\sqrt{d^2-d}=\left[d-1,\overline{2,2d-2}\right], d\in\mathbb{N}, d\geq 2;$
-

$$\alpha = \sqrt{d^2-d}=\frac{0+\sqrt{d^2-d}}{1}\Longrightarrow$$

$$s_0 = \mathbf{0},\ \ t_0=\mathbf{1},\ \ 1\Big|\Big(\Big(d^2-d\Big)-0^2\Big)$$

-

$$a_0=\Big\lfloor\alpha_0\Big\rfloor=\Big\lfloor\frac{s_0+\sqrt{d^2-d}}{t_0}\Big\rfloor=\Big\lfloor\frac{0+\sqrt{d^2-d}}{1}\Big\rfloor$$

$$\alpha_0=\sqrt{d^2-d}=\sqrt{d\,(d-1)}$$

$$d-1=\sqrt{(d-1)^2}<\ \sqrt{d\,(d-1)}<\sqrt{d^2}=d\Longrightarrow$$

$$a_0=\Big\lfloor\alpha_0\Big\rfloor=d-1$$

$$s_1=a_0t_0-s_0=(d-1)\cdot 1-0=\mathbf{d}-\mathbf{1},$$

$$t_1=\frac{d\,(d-1)-s_1^2}{t_0}=\frac{d\,(d-1)-(d-1)^2}{1}=\mathbf{d}-\mathbf{1}$$

- $$a_1 = \left\lfloor \alpha_1 \right\rfloor = \left\lfloor \frac{s_1 + \sqrt{d^2 - d}}{t_1} \right\rfloor = \left\lfloor \frac{d - 1 + \sqrt{d^2 - d}}{d - 1} \right\rfloor$$

$$\alpha_1 = \frac{d - 1 + \sqrt{d^2 - d}}{d - 1} = 1 + \frac{\sqrt{d\left(d - 1\right)}}{d - 1} = 1 + \sqrt{\frac{d}{d - 1}} =$$

$$= 1 + \sqrt{1 + \frac{1}{d - 1}} \implies 2 < \alpha_1 < 3$$

$$a_1 = \left\lfloor \alpha_1 \right\rfloor = 2$$

$$s_2 = a_1 t_1 - s_1 = 2 \cdot \left(d - 1\right) - \left(d - 1\right) = \mathbf{d - 1},$$

$$t_2 = \frac{d\left(d - 1\right) - s_2^2}{t_1} = \frac{d\left(d - 1\right) - \left(d - 1\right)^2}{d - 1} = \mathbf{1}$$

- $$a_2 = \left\lfloor \alpha_2 \right\rfloor = \left\lfloor \frac{s_2 + \sqrt{d^2 - d}}{t_2} \right\rfloor = \left\lfloor \frac{d - 1 + \sqrt{d^2 - d}}{1} \right\rfloor$$

$$\alpha_2 = d - 1 + \sqrt{d^2 - d} = d - 1 + \sqrt{d(d-1)}$$

$$2d - 2 = d - 1 + \sqrt{(d-1)^2} < d - 1 + \sqrt{d(d-1)}$$

$$d - 1 + \sqrt{d(d-1)} < d - 1 + \sqrt{d^2} = 2d - 1$$

$$\implies a_2 = \left\lfloor \alpha_2 \right\rfloor = 2d - 2$$

$$s_3 = a_2 t_2 - s_2 = (2d-2) \cdot 1 - (d-1) = \mathbf{d - 1},$$

$$t_3 = \frac{d(d-1) - s_3^2}{t_2} = \frac{d(d-1) - (d-1)^2}{1} = \mathbf{d - 1}$$

- $$(s_1, t_1) = (s_3, t_3) = (\mathbf{d-1}, \mathbf{d-1}) \implies j = 1,\ k = 3$$

$$\overset{r=2-\text{period}}{\Longrightarrow} \quad \alpha = [a_0, \ldots, a_{j-1}, \overline{a_j, \ldots, a_{k-1}}] =$$

$$= [a_0, \overline{a_1, a_2}] = \left[\mathbf{d-1}, \overline{\mathbf{2}, \mathbf{2d-2}}\right].$$

**Napomena:**

- $\alpha = \frac{2+\sqrt{5}}{3} = [\overline{1,2,2,1,12}]$ je reducirana kvadratna iracionalnost:

$$\alpha = \frac{2+\sqrt{5}}{3} \simeq 1.412 > 1$$
$$\alpha' = \frac{2-\sqrt{5}}{3} \simeq -0.0787 \Longrightarrow -1 < \alpha' < 0$$

$\overset{\text{Tr. 6.11}}{\Longrightarrow}$ čisto periodski razvoj.

- $\alpha = \sqrt{6} = [2, \overline{2,4}]$

Teorem 6.12:

$$\sqrt{d} = [a_0; \overline{a_1, a_2 ..., a_{r-1}, 2a_0}]$$

gdje je $a_0 = \left\lfloor \sqrt{d} \right\rfloor$, a niz $a_1, a_2 ..., a_{r-1}$ je centralno simetričan.

$$\sqrt{6}: \; a_0 = \left\lfloor \sqrt{6} \right\rfloor = 2, \;\; r = 2 \;\Longrightarrow\; [a_0; \overline{a_1, 2a_0}]$$

**Zadatak 6.5** Nađite jednostavni verižni (neprekidni) razlomak od:

**2.** $\sqrt{29} = \left[5, \overline{2, 1, 1, 2, 10}\right]$ .

- $$\alpha = \sqrt{29} \Longrightarrow\ s_0 = \mathbf{0},\ t_0 = \mathbf{1},\ a_0 = \left\lfloor \sqrt{29} \right\rfloor \overset{\alpha_0 \simeq 5.4}{=} \mathbf{5}$$

- $$s_1 = a_0 t_0 - s_0 = 5 \cdot 1 - 0 = \mathbf{5},\ t_1 = \frac{29 - s_1^2}{t_0} = \frac{29 - 5^2}{1} = \mathbf{4}$$

$$a_1 = \left\lfloor \alpha_1 \right\rfloor = \left\lfloor \frac{s_1 + \sqrt{29}}{t_1} \right\rfloor = \left\lfloor \frac{5 + \sqrt{29}}{4} \right\rfloor \overset{\alpha_1 \simeq 2.6}{=} \mathbf{2},$$

- $$s_2 = a_1 t_1 - s_1 = 2 \cdot 4 - 5 = \mathbf{3},\ t_2 = \frac{29 - s_2^2}{t_1} = \frac{29 - 3^2}{4} = \mathbf{5}$$

$$a_2 = \left\lfloor \alpha_2 \right\rfloor = \left\lfloor \frac{s_2 + \sqrt{29}}{t_2} \right\rfloor = \left\lfloor \frac{3 + \sqrt{29}}{5} \right\rfloor \overset{\alpha_0 \simeq 1.7}{=} \mathbf{1}$$

- $$s_3 = a_2 t_2 - s_2 = 1 \cdot 5 - 3 = \mathbf{2},\ \ t_3 = \frac{29 - s_3^2}{t_2} = \frac{29 - 3^2}{4} = \mathbf{5}$$

$$\Longrightarrow t_2 = t_3 = 5 \overset{n=2}{\Longrightarrow} r = 2n + 1 = 2 \cdot 2 + 1 = 5$$

- $$\sqrt{29} = [a_0; \_, \_, \_, \_, \_] \quad \Longrightarrow \quad \left[a_0; \overline{a_1, a_2, a_2, a_1, 2a_0}\right] \quad \Longrightarrow$$

$$\sqrt{29} = \left[5; \overline{2, 1, 1, 2, 10}\right]$$

- **Napomena:** $\alpha = \sqrt{d^2 - d}, d \in \mathbb{N}, d \geq 2;$

$$\alpha = \sqrt{d^2 - d} \Longrightarrow$$

$$s_0 = \mathbf{0},\ \ t_0 = \mathbf{1},\ \ a_0 = \left\lfloor \alpha_0 \right\rfloor = d - 1$$

$$s_1 = \mathbf{d - 1},\ \ t_1 = \mathbf{d - 1},\ \ a_1 = \left\lfloor \alpha_0 \right\rfloor = 2$$

$$s_2 = \mathbf{d - 1} \Longrightarrow s_1 = s_2 = d - 1 \overset{n=1}{\Longrightarrow} r = 2n = 2 \cdot 1 = 2$$

$$\sqrt{d^2 - d} = [a_0; \_, \_, ] \quad \Longrightarrow \quad \left[a_0; \overline{a_1, 2a_0}\right] \quad \Longrightarrow$$

$$\sqrt{d^2 - d} = \left[d - 1, \overline{2, 2d - 2}\right]$$

**Primjer** Nađite sve Pitagorine trokute u kojima je jedna stranica jednaka $39$.

<u>Rješenje:</u>

$$(x, y, z) = (dx_1, dy_1, dz_1) \quad \text{i} \quad (x_1, y_1, z_1) \ \text{primitivna}$$

$$\left[d\left(m^2 - n^2\right)\right]^2 + (d \cdot 2mn)^2 = \left[d\left(m^2 + n^2\right)\right]^2.$$

$$(x_1, y_1, z_1) = \left(m^2 - n^2,\ 2mn,\ m^2 + n^2\right)$$

$$39 = 3 \cdot 13 \overset{\tau(39)=4}{\Longrightarrow} d = 1, 3, 13, 39$$

**i)** $\mathbf{d = 1}$ (primitivne)

**1)** $m^2 + n^2 = 39 \implies$ (npr. $m-$paran) $n^2 = 3 \bmod 4$ $\implies$ nema;

**2)** $m^2 - n^2 = (m - n)(m + n) = 39$

$$39 = 39 \cdot 1 = 13 \cdot 3$$

**2a)** $m + n = 39, m - n = 1 \Longrightarrow m = 20, n = 19 \Longrightarrow$

$$\begin{aligned} z &= 20^2 + 19^2 = 761 \\ x &= 20^2 - 19^2 = 39 \\ y &= 2 \cdot 20 \cdot 19 = 760 \end{aligned}$$

$$\Longrightarrow (x, y, z) = (\mathbf{39}, \mathbf{760}, \mathbf{761})$$

Provjera: $39^2 + 760^2 = 579\,121 = 761^2$.

**2b)** $m + n = 13, m - n = 3 \Longrightarrow m = 8, n = 5 \Longrightarrow$

$$\begin{aligned} z &= 8^2 + 5^2 = 89 \\ x &= 8^2 - 5^2 = 39 \\ y &= 2 \cdot 8 \cdot 5 = 80 \end{aligned}$$

$$\Longrightarrow (x, y, z) = (\mathbf{39}, \mathbf{80}, \mathbf{89})$$

Provjera: $39^2 + 80^2 = 7921 = 89^2$.

**i)** $\mathbf{d = 3}$

**3)** $m^2 + n^2 = 13 \;\Longrightarrow\;$ (npr. $m-$paran) $n^2 = 1 \bmod 4$ $\Longrightarrow$ dobro;

$m^2 + n^2 = 13 \overset{m>n}{\Longrightarrow} 2n^2 \leq 13 \Longrightarrow$

$n \leq \sqrt{\frac{13}{2}} = 2.\,549\,5 \Longrightarrow n \leq 2 \Longrightarrow$

$$\begin{aligned} m^2 &= 13 - n^2, n = 1, 2. \\ m^2 &= 13 - 1^2 = 12 \Longrightarrow \text{nema} \\ m^2 &= 13 - 2^2 = 9 \Longrightarrow m = 3, n = 2 \end{aligned}$$

$$\begin{aligned} z_1 &= 3^2 + 2^2 = 13 \\ x_1 &= 3^2 - 2^2 = 5 \\ y_1 &= 2 \cdot 3 \cdot 2 = 12 \end{aligned}$$

$$\Longrightarrow (x, y, z) = (3x_1, 3y_1, 3z_1) = (\mathbf{15}, \mathbf{36}, \mathbf{39}) \,.$$

Provjera: $15^2 + 36^2 = 1521 = 39^2$.

**4)** $m^2 - n^2 = (m - n)(m + n) = 13 \Longrightarrow 13 = 13 \cdot 1$
$m + n = 13,\, m - n = 1 \Longrightarrow m = 7, n = 6 \Longrightarrow$

$$\begin{aligned} z_1 &= 7^2 + 6^2 = 85 \\ x_1 &= 7^2 - 6^2 = 13 \\ y_1 &= 2 \cdot 7 \cdot 6 = 84 \end{aligned}$$

$$\Longrightarrow (x, y, z) = (3x_1, 3y_1, 3z_1) = (\mathbf{39}, \mathbf{252}, \mathbf{255}) \,.$$

Provjera: $39^2 + 252^2 = 65\,025 = 255^2$.

**iii)** $\mathbf{d = 13}$

**5)** $m^2 + n^2 = 3 \;\Longrightarrow\;$ (npr. $m-$paran) $n^2 = 3 \bmod 4$
$\Longrightarrow$ nema;

**6)** $m^2 - n^2 = (m-n)(m+n) = 3 \Longrightarrow 3 = 3 \cdot 1$
$m+n = 3,\, m-n = 1 \Longrightarrow m = 2, n = 1 \Longrightarrow$

$$\begin{aligned} z_1 &= 2^2 + 2^2 = 8 \\ x_1 &= 2^2 - 1^2 = 3 \\ y_1 &= 2 \cdot 2 \cdot 1 = 4 \end{aligned}$$

$$\Longrightarrow (x, y, z) = (13x_1, 13y_1, 13z_1) = (\mathbf{39}, \mathbf{52}, \mathbf{65}).$$

Provjera: $39^2 + 52^2 = 4225 = 65^2$.

$\mathbf{d} = \mathbf{39}$ (nepotrebno)

**7)** $m^2 + n^2 = 1 \Longrightarrow m > n \Longrightarrow m = 1, n = 0 \Longrightarrow$
nema;

**8)** $m^2 - n^2 = (m-n)(m+n) = 1 \Longrightarrow 1 = 1 \cdot 1$
$m+n = 1,\, m-n = 1 \Longrightarrow m = 1, n = 0 \Longrightarrow$ nema;

**Primjer** Nađite sve Pitagorine trokute u kojima je jedna stranica jednaka $12$.

Rješenje:

$$(x, y, z) = (dx_1, dy_1, dz_1) \quad \text{i} \quad (x_1, y_1, z_1) \text{ primitivna}$$

$$\left[d\left(m^2 - n^2\right)\right]^2 + (d \cdot 2mn)^2 = \left[d\left(m^2 + n^2\right)\right]^2.$$

$$(x_1, y_1, z_1) = \left(m^2 - n^2,\ 2mn,\ m^2 + n^2\right)$$

$$12 = 3 \cdot 2^2 \stackrel{\tau(12)=6}{\Longrightarrow} d = 1, 2, 4, 3, 6, 12$$

**i)** $\mathbf{d = 1}$ (primitivne)

**1)** $2mn = 12 \Longrightarrow mn = 6 \Longrightarrow m = 6, n = 1 \ \ i$ $m = 3, n = 2 \Longrightarrow$

**1a)**

$$\begin{aligned} z &= 6^2 + 1^2 = 37 \\ x &= 6^2 - 1^2 = 35 \\ y &= 2 \cdot 6 \cdot 1 = 12 \end{aligned}$$

$$\Longrightarrow (x, y, z) = (\mathbf{35}, \mathbf{12}, \mathbf{37}).$$

Provjera: $35^2 + 12^2 = 1369 = 37^2$.

**1b)**

$$\begin{aligned} z &= 3^2 + 2^2 = 13 \\ x &= 3^2 - 2^2 = 5 \\ y &= 2 \cdot 3 \cdot 2 = 12 \end{aligned}$$

$$\Longrightarrow (x, y, z) = (\mathbf{5}, \mathbf{12}, \mathbf{13})\,.$$

Provjera: $5^2 + 12^2 = 169 = 13^2$.

**ii)** $\mathbf{d = 2}$

**2)** $2mn = 6 \implies mn = 3 \Longrightarrow m = 3, n = 1$ ništa - iste su parnosti;

**iii)** $\mathbf{d = 4}$

**3)** $m^2 - n^2 = (m-n)(m+n) = 3 = 3 \cdot 1 \Longrightarrow m + n = 3, m - n = 1 \Longrightarrow m = 2, n = 1$

$$\begin{aligned} z &= 2^2 + 1^2 = 5 \\ x &= 2^2 - 1^2 = 3 \\ y &= 2 \cdot 2 \cdot 1 = 4 \end{aligned}$$

$$\Longrightarrow (x, y, z) = (4x_1, 4y_1, 4z_1) = (\mathbf{12}, \mathbf{16}, \mathbf{20})\,.$$

Provjera: $12^2 + 16^2 = 400 = 20^2$.

**4)** $m^2 + n^2 = 3 \Longrightarrow 2n^2 \leq 3 \Longrightarrow n^2 = 1 \Longrightarrow m^2 = 2 \Longrightarrow$ništa

**iv)** $\mathbf{d = 3}$

**5)** $2mn = 4 \Longrightarrow mn = 2 \Longrightarrow m = 2, n = 1 \Longrightarrow$

$$\begin{aligned} z &= 2^2 + 1^2 = 5 \\ x &= 2^2 - 1^2 = 3 \\ y &= 2 \cdot 2 \cdot 1 = 4 \end{aligned}$$

$$\Longrightarrow (x, y, z) = (3x_1, 3y_1, 3z_1) = (\mathbf{9}, \mathbf{12}, \mathbf{15}).$$

Provjera: $9^2 + 12^2 = 225 = 15^2$.

**v)** $\mathbf{d = 6}$

**6)** $2mn = 2 \Longrightarrow mn = 1 \Longrightarrow m = 1, n = 1 \Longrightarrow$ ništa iste su parnosti i $m = n$;

**Primjer** Nađite sve Pitagorine trokute u kojima je jedna stranica jednaka $63$.

<u>Rješenje:</u>

$$(x, y, z) = (dx_1, dy_1, dz_1) \quad \text{i} \quad (x_1, y_1, z_1) \text{ primitivna}$$

$$\left[d\left(m^2 - n^2\right)\right]^2 + (d \cdot 2mn)^2 = \left[d\left(m^2 + n^2\right)\right]^2.$$

$$(x_1, y_1, z_1) = \left(m^2 - n^2,\ 2mn,\ m^2 + n^2\right)$$

$$63 = 3^2 7 \overset{\tau(63)=6}{\Longrightarrow} d = 1, 3, 9, 7, 21, 63$$

**i)** $\mathbf{d = 1}$ (primitivne)

**1)** $m^2 + n^2 = 63 \implies$ (npr. $m-$paran) $n^2 = 3 \bmod 4$ $\implies$ nema;

**2)** $m^2 - n^2 = (m - n)(m + n) = 63$

$$63 = 63 \cdot 1 = 21 \cdot 3 = 9 \cdot 7$$

**2a)** $m + n = 63, m - n = 1 \Longrightarrow m = 32, n = 31 \Longrightarrow$

$$\begin{aligned} z &= 32^2 + 31^2 = 1985 \\ x &= 32^2 - 31^2 = 63 \\ y &= 2 \cdot 32 \cdot 31 = 1984 \end{aligned}$$

$$\Longrightarrow (x, y, z) = (\mathbf{63}, \mathbf{1984}, \mathbf{1985})$$

Provjera: $63^2 + 1984^2 = 3940\,225 = 1985^2$

**2b)** $m + n = 21,\ m - n = 3 \Longrightarrow m = 12, n = 9 \Longrightarrow$ ništa - nisu relativno prosti. Imali bi:

$$\begin{aligned} z &= 12^2 + 9^2 = 225 \\ x &= 12^2 - 9^2 = 63 \\ y &= 2 \cdot 12 \cdot 9 = 216 \end{aligned}$$

$$\Longrightarrow (x, y, z) = (\mathbf{63}, \mathbf{216}, \mathbf{225})$$

Ova trojka nije primitivna - pojavit će se kasnije.

**2c)** $m + n = 9,\ m - n = 7 \Longrightarrow m = 8, n = 1 \Longrightarrow$

$$\begin{aligned} z &= 8^2 + 1^2 = 65 \\ x &= 8^2 - 1^2 = 63 \\ y &= 2 \cdot 8 \cdot 1 = 16 \end{aligned}$$

$$\Longrightarrow (x, y, z) = (\mathbf{63}, \mathbf{16}, \mathbf{65})\,.$$

Provjera: $63^2 + 16^2 = 4225 = 65^2$.

**ii)** $\mathbf{d = 3}$

**1a)** $m^2+n^2=21 \implies$ (npr. $m-$paran) $n^2=1 \bmod 4$ $\implies$ dobro;

$$2n^2 \le m^2+n^2 = 21 \implies n^2 \le \tfrac{21}{2} = 10.5$$

$\implies \quad n \le 3 \implies \quad n = 1, 2, 3.$

$m^2 = 21 - 1 = 20 \implies$ ništa;

$m^2 = 21 - 4 = 17, \implies$ ništa;

$m^2 = 21 - 9 = 12 \implies$ ništa.

**2)** $m^2 - n^2 = (m-n)(m+n) = 21$

$$21 = 21 \cdot 1 = 7 \cdot 3$$

**2a)** $m+n=21, m-n=1 \implies m=11, n=10 \implies$

$$\begin{aligned} z_1 &= 11^2 + 10^2 = 221 \\ x_1 &= 11^2 - 10^2 = 21 \\ y_1 &= 2 \cdot 11 \cdot 10 = 220 \end{aligned}$$

$$\implies (x, y, z) = (3x_1, 3y_1, 3z_1) = (\mathbf{63}, \mathbf{660}, \mathbf{663})$$

Provjera: $63^2 + 660^2 = 439\,569 = 663^2$

**2b)** $m+n=7, m-n=3 \Longrightarrow m=5, n=2 \Longrightarrow$

$$\begin{aligned} z_1 &= 5^2+2^2=29 \\ x_1 &= 5^2-2^2=21 \\ y_1 &= 2\cdot 5\cdot 2=20 \end{aligned}$$

$$\Longrightarrow (x,y,z)=(3x_1,3y_1,3z_1)=(\mathbf{63},\mathbf{60},\mathbf{87})$$

Provjera: $63^2+60^2=7569=87^2$

**iii)** $\mathbf{d=9}$

**1a)** $m^2+n^2=7 \;\Longrightarrow\;$ (npr. $m-$paran) $n^2=3 \bmod 4$ $\Longrightarrow$ ništa;

**2)** $m^2-n^2=(m-n)(m+n)=7$

$$7=7\cdot 1$$

**2a)** $m+n=7, m-n=1 \Longrightarrow m=4, n=3 \Longrightarrow$

$$\begin{aligned} z_1 &= 4^2+3^2=25 \\ x_1 &= 4^2-3^2=7 \\ y_1 &= 2\cdot 4\cdot 3=24 \end{aligned}$$

$\Longrightarrow (x,y,z) = (9x_1, 9y_1, 9z_1) = (\mathbf{63}, \mathbf{216}, \mathbf{225})$ **(!)**

Provjera: $63^2 + 216^2 = 50\,625 = 225^2$

**iv)** $\mathbf{d = 7}$

**1a)** $m^2 + n^2 = 9 \implies$ (npr. $m-$paran) $n^2 = 1 \bmod 4$ $\Longrightarrow$dobro;

$$2n^2 \leq m^2 + n^2 = 9 \implies n^2 \leq \frac{9}{2} = 4.5$$

$\Longrightarrow \quad n \leq 2 \Longrightarrow \quad n = 1, 2.$

$m^2 = 9 - 1 = 8 \Longrightarrow$ ništa;

$m^2 = 9 - 5 = 5 \Longrightarrow$ ništa.

**2)** $m^2 - n^2 = (m-n)(m+n) = 9$

$$9 = 9 \cdot 1 = 3 \cdot 3$$

**2a)** $m + n = 9, m - n = 1 \Longrightarrow m = 5, n = 4 \Longrightarrow$

$$\begin{aligned} z_1 &= 5^2 + 4^2 = 41 \\ x_1 &= 5^2 - 4^2 = 9 \\ y_1 &= 2 \cdot 5 \cdot 4 = 40 \end{aligned}$$

$$\Longrightarrow (x, y, z) = (7x_1, 7y_1, 7z_1) = (\mathbf{63}, \mathbf{280}, \mathbf{287})$$

Provjera: $63^2 + 280^2 = 82\,369 = 287^2$

**2b)** $m + n = 3,\ m - n = 3 \Longrightarrow m = 3, n = 0 \Longrightarrow$ ništa

**v)** $\mathbf{d = 21}$

**1a)** $m^2 + n^2 = 3 \implies$ (npr. $m-$paran) $n^2 = 3 \bmod 4$
$\Longrightarrow$ ništa;

**2)** $m^2 - n^2 = (m - n)(m + n) = 3$

$$3 = 3 \cdot 1$$

**2a)** $m + n = 3,\ m - n = 1 \Longrightarrow m = 2, n = 1 \Longrightarrow$

$$\begin{aligned} z_1 &= 2^2 + 1^2 = 5 \\ x_1 &= 2^2 - 1^2 = 3 \\ y_1 &= 2 \cdot 2 \cdot 1 = 4 \end{aligned}$$

$$\Longrightarrow (x, y, z) = (21x_1, 21y_1, 21z_1) = (\mathbf{63}, \mathbf{84}, \mathbf{105})$$

Provjera: $63^2 + 84^2 = 11\,025 = 105^2$

**Primjer** Nađite najmanje rješenje u prirodnim brojevima Pellovih jednadžbi $x^2 - 47y^2 = 1$ i $x^2 - 47y^2 = -1$ (ako postoje).

Rješenje:

- $$\alpha = \sqrt{47} = \frac{0+\sqrt{47}}{1} \Longrightarrow \mathbf{s}_0 = \mathbf{0},\ \mathbf{t}_0 = \mathbf{1},\ 1 \,\Big|\, \left(47 - 0^2\right) \Longrightarrow$$

$$a_0 = \lfloor \alpha_0 \rfloor = \lfloor \sqrt{47} \rfloor \overset{\alpha_0 \simeq 6.85}{=} \mathbf{6},$$

$$s_1 = a_0 t_0 - s_0 = 6 \cdot 1 - 0 = \mathbf{6},\ \ t_1 = \frac{47 - s_1^2}{t_0} = \frac{47 - 6^2}{1} = \mathbf{11}$$

- $$a_1 = \lfloor \alpha_1 \rfloor = \left\lfloor \frac{s_1 + \sqrt{47}}{t_1} \right\rfloor = \left\lfloor \frac{6+\sqrt{47}}{11} \right\rfloor \overset{\alpha_1 \simeq 1.2}{=} \mathbf{1},$$

$$s_2 = a_1 t_1 - s_1 = 1 \cdot 11 - 6 = \mathbf{5},\ \ t_2 = \frac{47 - s_2^2}{t_1} = \frac{47 - 5^2}{11} = \mathbf{2}$$

- $$a_2 = \left\lfloor \alpha_2 \right\rfloor = \left\lfloor \frac{s_2+\sqrt{47}}{t_2} \right\rfloor = \left\lfloor \frac{5+\sqrt{47}}{2} \right\rfloor \overset{\alpha_2 \simeq 5.9}{=} \mathbf{5},$$

  $$s_3 = a_2 t_2 - s_2 = 5 \cdot 2 - 5 = \mathbf{5}, \;\; t_3 = \frac{47 - s_3^2}{t_2} = \frac{47 - 5^2}{2} = \mathbf{11}$$

- $$\Longrightarrow s_2 = s_3 = 5 \overset{n=2}{\Longrightarrow} \mathbf{r} = 2n = 2 \cdot 2 = \mathbf{4}$$

  $$\sqrt{47} = [a_0; \_, \_, \_, \_,] \quad \Longrightarrow \quad [a_0; \overline{a_1, a_2, a_1, 2a_0}] \quad \Longrightarrow$$

  $$\sqrt{47} = [6, \overline{1, 5, 1, 12}]$$

- Kako je $r = 4$ paran, onda jednadžba $x^2 - 47y^2 = -1$ nema rješenja, a sva rješenja u prirodnim brojevima od $x^2 - 47y^2 = 1$ su dana sa $x = p_{4k-1}$, $y = q_{4k-1}$, $k \in \mathbb{N}$.

- Najmanje $(x, y) = (p_3, q_3)$;

- Traženje konvergenti, tj. rješenja u prirodnim brojevima:

$$p_n = a_n p_{n-1} + p_{n-2}, \qquad p_0 = a_0, \quad p_1 = a_0 a_1 + 1;$$
$$q_n = a_n q_{n-1} + q_{n-2}, \qquad q_0 = 1, \quad q_1 = a_1.$$

| $n$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $a_n$ | 6 | 1 | 5 | 1 |
| $p_n$ | 6 | 7 | 41 | **48** |
| $q_n$ | 1 | 1 | 6 | **7** |

- Najmanje $(x, y) = (\mathbf{48}, \mathbf{7})$ .