Determining elements of minimal index in an infinite family of totally real bicyclic biquadratic number fields

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Abstract

Let $c \neq 2$ be a positive integer such that c and $c+4$ are square-free. We consider the infinite parametric family of bicyclic biquadratic

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fields $K = \mathbb{Q}(\sqrt{2c}, \sqrt{2(c+4)})$. We determine the integral basis of the field. We show that K admits no power integral basis, determine the minimal index and all elements of minimal index. We use the solutions of a parametric family of quartic Thue equations and extensive numerical calculations by Maple and Magma are also involved.

1 Introduction

It is a classical problem in algebraic number theory (cf. [3]) to decide if a number field K is *monogene*, that is if there exist an $\alpha \in \mathbb{Z}_K$ such that $\mathbb{Z}_K = \mathbb{Z}[\alpha]$, where $\mathbb{Z}_K$ is the ring of integers of K see [14], [3]. In this case $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is a *power integral basis* of K.

The *index* $I(\alpha)$ of $\alpha \in \mathbb{Z}_K$ is defined by

$$I(\alpha) = (\mathbb{Z}_K^+ : \mathbb{Z}[\alpha]^+)$$

where the index of the additive groups of the above rings is taken. Obviously, α generates a power integral basis if and only if $I(\alpha) = 1$. The *minimal index* m_K of the field K is determined as the minimum of all $I(\alpha)$ taken for all $\alpha \in \mathbb{Z}_K$ with $K = \mathbb{Q}(\alpha)$ (primitive elements).

If α has index m, then any $\beta = a \pm \alpha$ also has index m with any $a \in \mathbb{Z}$. Such algebraic integers are called *equivalent* and we determine elements of given index up to equivalence.

In case of quartic fields there exists a general method for determining power integral bases and elements of given index [5], [7], [3] which made possible also to investigate infinite parametric families of quartic fields, cf. [8]. Especially bicyclic biquadratic number fields of type $K = \mathbb{Q}(\sqrt{m}, \sqrt{n})$ have an extensive literature, see [6]. These fields are better known in the complex case [15]. The totally real fields, like those we investigate in this paper, seem to be much more interesting.

In our paper we consider the infinite parametric family

$$K = \mathbb{Q}(\sqrt{2c}, \sqrt{2(c+4)})$$

of bicyclic biquadratic number fields. We prove that it admits no power integral bases. We calculate the minimal index and all elements of minimal index.

In our arguments the result of Bo He, B.Jadrijević, A.Togbé [9] on the solutions of the parametric family of Thue equations

$$|x^4-4x^3y+(2-2c)x^2y^2+(4c+4)xy^3+(1-2c)y^4| \leq \max\left\{\frac{c}{4}, 4\right\} \quad (\text{in } x, y \in \mathbb{Z})$$

plays an essential role. We also make use of an extensive calculation by Maple and Magma involving the complete resolution of thousands of Thue equations.

Recently B.Jadriević investigated infinite parametric families of totally real bicyclic biquadratic number fields [10], [11], [12], determining monogenity and elements of minimal index reducing the problem to a system of Pellian equations. Here we consider a similar type of family of number fields but we use a quite different technics, involving extensive formal and numerical calculations, as well. These arguments might be useful also in other families of number fields.

2 Integral basis, discriminant

In our paper we shall deal with the number field generated by a root of the following polynomial.

Lemma 1. *Let c be an integer. Then the polynomial $f(x) = x^4-(2c+4)x^2+4$ is reducible if and only if $c = 2a_0^2 - 4$ or $c = 2a_0^2$ for $a_0 \in \mathbb{Z}$.*

Proof. Assume that

$$x^4 - (2c+4)x^2 + 4 = (x-m)p(x)$$

with some $m \in \mathbb{Z}$. Then

$$m = \pm\sqrt{c - \sqrt{c(c+4)} + 2} \;\text{ or }\; m = \pm\sqrt{c + \sqrt{c(c+4)} + 2},$$

Since $m \in \mathbb{Z}$, we have

$$c \pm \sqrt{c(c+4)} + 2 = a^2,\; a \in \mathbb{Z} \text{ and } b = \sqrt{c(c+4)} \in \mathbb{Z}.$$

Hence

$$\pm b = a^2 - c - 2 \text{ and } b^2 = c(c+4)$$

or

$$c(c+4) = (-a^2 + c + 2)^2,$$

which implies

$$c = \frac{(a^2-2)^2}{2a^2} = \frac{2}{a^2} + \frac{1}{2}a^2 - 2 \notin \mathbb{Z} \text{ for } a \neq 0,$$

a contradiction. The equation $c(c+4) = (-a^2 + c + 2)^2$ has no solutions in c for $a = 0$, either.

If

$$x^4 - (2c+4)x^2 + 4 = (x^2 + ax + d)(x^2 + bx + e)$$

with some $a, b, d, e \in \mathbb{Z}$, then $(d, e) = \pm(2, 2), \pm(1, 4)$. By checking all possibilities we find that $(d, e) = \pm(2, 2)$ and $a = -b$. This implies

$$x^4 - (2c+4)x^2 + 4 = (x^2 + ax + 2)(x^2 - ax + 2) \text{ whence } c = \frac{1}{2}a^2 - 4$$

or

$$x^4 - (2c+4)x^2 + 4 = (x^2 + ax - 2)(x^2 - ax - 2) \text{ whence } c = \frac{1}{2}a^2$$

and a is even, $a = 2a_0$. Therefore

$$c + 4 = 2a_0^2 \text{ or } c = 2a_0^2 \text{ where } a_0 \in \mathbb{Z}.$$

□

Corollary 2. *Let $c \neq 2$ be a positive integer such that c and $c+4$ are squarefree. Then $f(x) = x^4 - (2c+4)x^2 + 4$ is irreducible.*

Throughout the paper we assume that $c \neq 2$ is a positive integer, such that that c and $c+4$ are squarefree. Note that $\gcd(c, c+4) = 1$ if c is odd and $\gcd(c, c+4) = 2$ if c is even.

Lemma 3. *The field $K = \mathbb{Q}(\sqrt{2c}, \sqrt{2(c+4)})$ is generated by the root ξ of the polynomial*

$$f(x) = x^4 - (2c+4)x^2 + 4.$$

Proof. Set

$$\alpha^{(1)} = \sqrt{2c}, \alpha^{(2)} = \sqrt{2c}, \alpha^{(3)} = -\sqrt{2c}, \alpha^{(4)} = -\sqrt{2c},$$

$$\beta^{(1)} = \sqrt{2(c+4)}, \beta^{(2)} = \sqrt{2(c+4)}, \beta^{(3)} = -\sqrt{2(c+4)}, \beta^{(4)} = -\sqrt{2(c+4)}.$$

The element $\sqrt{2c} + \sqrt{2(c+4)}$ is obviously contained in K, hence it is a generating element of K. The defining polynomial of $\sqrt{2c} + \sqrt{2(c+4)}$ is

$$g(x) = \prod_{i=1}^{4} (x - \alpha^{(i)} - \beta^{(i)}) = x^4 - (8c+16)x^2 + 64.$$

We have $g(2x) = 16(x^4 - (2c+4)x^2 + 4)$, hence the integral element ξ, being a root of $f(x)$ generates the same field K. □

Corollary 4. *Let $c \neq 2$ be a positive integer such that c and $c+4$ are squarefree. Then $K = \mathbb{Q}(\sqrt{2c}, \sqrt{2(c+4)})$ is a quartic field.*

Proof. Straightforward from the above statements. □

Remark Note that if $c \neq 2$ is even such that c and $c+4$ are squarefree integers, then $c \equiv 2 \pmod 4$, ie. $c = 2c_0$ where $c_0 > 1$ is odd and $K = \mathbb{Q}(\sqrt{2c}, \sqrt{2(c+4)}) = \mathbb{Q}(\sqrt{c_0}, \sqrt{c_0+2})$.

Theorem 5. *Let ξ be a root of the polynomial*

$$f(x) = x^4 - (2c+4)x^2 + 4.$$

An integral basis of K is given by

$$\begin{aligned} &\left\{1, \xi, \frac{\xi^2}{2}, \frac{\xi^3}{2}\right\}, && \text{if } c \text{ is odd}, \\ &\left\{1, \xi, \frac{\xi^2+2\xi+2}{4}, \frac{\xi^3+2\xi+4}{8}\right\}, && \text{if } c \equiv 2 \pmod 8), \\ &\left\{1, \xi, \frac{\xi^2+2\xi+2}{4}, \frac{\xi^3+6\xi+4}{8}\right\}, && \text{if } c \equiv 6 \pmod 8). \end{aligned}$$

Proof. Let $c = 2^a c_0, c+4 = 2^b d_0$ where c_0, d_0 are odd. Then $(c_0, d_0) = 1$ since a common odd prime factor of $c, c+4$ is not possible. We have $a = b = 0$ if c is odd, $a = b = 1$ if c is even.

The discriminant of the polynomial $f(x)$, as well as the discriminant of the element ξ is

$$D(\xi) = 2^{10}c^2(c+4)^2 = 2^{10+4a}c_0^2 d_0^2.$$

The field K admits the quadratic subfields $M_1 = \mathbb{Q}(\sqrt{2c})$ and $M_2 = \mathbb{Q}(\sqrt{2(c+4)})$. By a well known theorem (see [14]) we have

$$D_K = N_{M_i/\mathbb{Q}}(D_{K/M_i}) \cdot D_{M_i}^2$$

where D_K and D_{M_i} denote the discriminants of K and M_i, respectively, and D_{K/M_i} is the relative discriminant ($i = 1, 2$). The discriminant of $M_1 = \mathbb{Q}(\sqrt{2c})$ is divisible by c_0^2, the discriminant of $M_2 = \mathbb{Q}(\sqrt{2(c+4)})$ is divisible by d_0^2. Hence, by $(c_0, d_0) = 1$ we have

$$D_K = 2^k c_0^2 d_0^2.$$

Therefore the discriminant of the basis $\{1, \xi, \xi^2, \xi^3\}$ can only be diminished by a 2-power factor to obtain an integral basis. Denote by b_1, b_2, b_3, b_4 the elements given in the bases of our theorem. Using symmetric polynomials it is easy to see that all these elements are algebraic integers. Moreover, consider the elements

$$\frac{\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 + \lambda_4 b_4}{2}$$

for $0 \leq \lambda_1, \lambda_2, \lambda_3, \lambda_4 \leq 1$. Again by symmetric polynomials we can see that none of these elements are algebraic integers. (For this purpose we use the properties of c modulo 4.) Hence the bases given in the Theorem are indeed integer bases. □

Corollary 6. *The discriminant of the field K is*

$$D_K = (8c^2(c+4))^2 \quad \text{if } c \text{ is odd,}$$
$$D_K = (c^2(c+4))^2 \quad \text{if } c \equiv 2, 6 \pmod 8.$$

Proof. Direct calculation using the integral bases. □

Remark An other type of integral basis and the discriminant of K can also be obtained by using the results of [6].

3 Elements of given index in quartic fields

In [5] a general algorithm is given for determining elements of given index in quartic fields. Here we shortly recall this result.

Let $K = \mathbb{Q}(\xi)$ be a quartic number field and let $f(x) = x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 \in \mathbb{Z}[x]$ the minimal polynomial of ξ. Assume that we can represent any integer in K in the form

$$\alpha = \frac{a + x\xi + y\xi^2 + z\xi^3}{d} \tag{1}$$

with $a, x, y, z \in \mathbb{Z}$ and with a common denominator $d \in \mathbb{Z}$. Let $d_1 = \gcd(x, y, z)$. Set $x_1 = x/d_1, y_1 = y/d_1, z_1 = z/d_1$.

Lemma 7. *The element α (with $a, x, y, z \in \mathbb{Z}$) has index m in $\mathbb{Z}_K$ if and only if there is a solution $(u, v) \in \mathbb{Z}^2$ of the cubic equation*

$$F(u, v) = u^3 - a_2u^2v + (a_1a_3 - 4a_4)uv^2 + (4a_2a_4 - a_3^2 - a_1^2a_4)v^3 = \pm\frac{d^6m}{n} \tag{2}$$

such that (x_1, y_1, z_1) satisfies

$$\begin{aligned} Q_1(x_1, y_1, z_1) &= x_1^2 - x_1y_1a_1 + y_1^2a_2 + x_1z_1(a_1^2 - 2a_2) + y_1z_1(a_3 - a_1a_2) \\ &\quad + z_1^2(-a_1a_3 + a_2^2 + a_4) = \frac{u}{d_1^2} \;, \\ Q_2(x_1, y_1, z_1) &= y_1^2 - x_1z_1 - a_1y_1z_1 + z_1^2a_2 = \frac{v}{d_1^2} \;. \end{aligned} \tag{3}$$

Remark If (x, y, z) is a suitable triple satisfying the conditions of the above Lemma, we also have to check if the corresponding α represented in the form (1) is indeed an algebraic integer in K and to calculate from (x, y, z) the coordinates (f_2, f_3, f_4) of the element in the integral basis (corresponding to the basis elements b_2, b_3, b_4 since we determine elements up to equivalence). According to Theorem 5 we must have

$$\begin{aligned} (f_2, f_3, f_4) &= \left(\frac{x}{2}, y, z\right), \text{ if } c \text{ is odd}, \\ (f_2, f_3, f_4) &= \left(\frac{x - 2y - 2z}{8}, \frac{y}{2}, z\right), \text{ if } c \equiv 2 \pmod 8, \\ (f_2, f_3, f_4) &= \left(\frac{x - 2y - 6z}{8}, \frac{y}{2}, z\right), \text{ if } c \equiv 6 \pmod 8. \end{aligned}$$

In order to have an algebraic integer element corresponding to (x, y, z) satisfying the above Lemma, the values of (f_2, f_3, f_4) must be integer.

4 Elements of minimal index in K

Denote by $\{b_1 = 1, b_2, b_3, b_4\}$ the integral basis of K according to Theorem 5. In our following statement we display the coordinates (f_2, f_3, f_4) corresponding to the basis elements (b_2, b_3, b_4) of the elements of given index. (We determine elements up to equivalence, see Introduction.)

The main result of our paper is the following.

Theorem 8.
There are no power integral bases in K.

If c is odd, then the minimal index of K is 4, and the only elements of minimal index are
$(f_2, f_3, f_4) = (1, 0, 0), (-c-2, 0, 1)$.
Further, for $c = 1$ the elements
$(f_2, f_3, f_4) = (-5, \pm 2, 2), (-1, 2, 1)$ *also have index 4.*

If c is even, $c \neq 6, 10, 22$, then the minimal index of K is 32, all elements of minimal index are
$(f_2, f_3, f_4) = (1, 0, 0)$, *further*
$(f_2, f_3, f_4) = (-c-3, 0, 4)$ *for* $c \equiv 2 \pmod 8$ *and*
$(f_2, f_3, f_4) = (-c-5, 0, 4)$ *for* $c \equiv 6 \pmod 8$.

If $c = 8l^2 - 2, (l \in \mathbb{N})$ *we have the additional solutions*
$(f_2, f_3, f_4) = (\pm 4l - 3, \pm 8l, 4), (-64l^4 - 24l^2 \pm 4l + 1, \pm 8l, 32l^2)$.

For $c = 6$ the minimal index of K is 4 and all elements of index 4 are
$(f_2, f_3, f_4) = (0, 1, 0), (6, -1, -2), (5, 1, -2), (1, -1, 0)$.
For $c = 10$ the minimal index of K is 23 and all elements of index 23 are
$(f_2, f_3, f_4) = (10, 0, -3), (-4, 0, 1), (10, 0, -3)$.
For $c = 22$ the minimal index of K is 31 and all elements of index 31 are
$(f_2, f_3, f_4) = (20, 0, -3), (-6, 0, 1), (140, 3, -21), (143, -3, -21)$
$(-3, 3, 1), (0, -3, 1), (20, 0, -3)$.

Proof. Set d, n and e according to the following table:

	d	n	$e = d^6/n$
c odd	2	4	2^4
$c \equiv 2, 6 \pmod 8$	8	32	2^{13}

Using the integral basis of Theorem 5 we can represent any $\alpha \in \mathbb{Z}_K$ in the form (1) using denominator d. The index of ξ is n. In order to determine elements of index m we have to solve (2) with right hand side $\pm e \cdot m$. Further, since the index of ξ is n, in order to determine the minimal index and all elements with minimal index we have to determine elements of index $m = 1, 2, \ldots, n$, until we find suitable elements. Note also that d_1^6 divides $e \cdot m = d^6 m/n$ (cf. [7]).

For the defining polynomial $f(x)$ of ξ we have $a_1 = 0, a_2 = -2c - 4, a_3 = 0, a_4 = 4$. The binary form $F(u, v)$ in (2) factorizes as

$$F(u, v) = (u - 4v)(u + 4v)(u + (2c + 4)v).$$

For elements of index m the right hand side is $\pm e \cdot m$. Therefore we consider the equation

$$(u - 4v)(u + 4v)(u + (2c + 4)v) = \pm e \cdot m \quad \text{in} \quad u, v \in \mathbb{Z}. \tag{4}$$

For $m = 1, 2, \ldots, n$ we calculated those integers α, β, γ with $\alpha\beta\gamma = \pm e \cdot m$. For each triplet we checked if the system of equations

$$\begin{aligned} u - 4v &= \alpha \\ u + 4v &= \beta \\ u + (2c + 4)v &= \gamma \end{aligned} \tag{5}$$

is solvable in integers u, v. The values of

$$u = \frac{\alpha + \beta}{2}, \quad v = \frac{\beta - \alpha}{8}$$

must be integers and the integer c must satisfy

$$\gamma - u = v(2c + 4).$$

Hence, either $v = 0$, $\gamma = u$ and c is arbitrary, or $v \neq 0$ and

$$c = \frac{\gamma - u - 4v}{2v}$$

is an integer.

A. The case of odd parameters c

Using a simple computer test we checked all cases and found that (4) has no solutions for $m = 1, 2, 3$, only for $m = 4$ and the only solution is $u = \pm 4$, $v = 0$.

For this pair (u, v) we have to solve the system of equations (3), that is

$$Q_1(x_1, y_1, z_1) = x_1^2 - y_1^2(2c+4) + x_1 z_1(4c+8) + z_1^2(4+(2c+4)^2) = \pm\frac{4}{d_1^2} \quad (6)$$

$$Q_2(x_1, y_1, z_1) = y_1^2 - x_1 z_1 - z_1^2(2c+4) = 0. \quad (7)$$

Since d_1^6 divides the right hand side of (4) (cf. [7]), i.e. $e \cdot m = 2^6$, hence we have $d_1 = 1$ or 2. Following the algorithm given in [7] we set

$$Q_0(x_1, y_1, z_1) = uQ_2(x_1, y_1, z_1) - vQ_1(x_1, y_1, z_1).$$

The equation $Q_0(x_1, y_1, z_1) = 0$ is equivalent to $Q_2(x_1, y_1, z_1) = 0$ that is

$$y_1^2 - x_1 z_1 - z_1^2(2c+4) = 0.$$

A nontrivial solution of this equation is $x_0 = 1, y_0 = 0, z_0 = 0$. We use the parametrization

$$x_1 = rx_0 = r, \; y_1 = ry_0 + p = p, \; z_1 = rz_0 + q = q \quad (8)$$

with rational parameters r, p, q. Substituting these into the above equation we get

$$rq = p^2 - (2c+4)q^2.$$

Multiplying equation (8) by q and using the above equation we obtain

$$\begin{array}{lcl} kx_1 & = & p^2 - (2c+4)q^2 \\ ky_1 & = & pq \\ kz_1 & = & q^2 \end{array} \quad (9)$$

Multiplying these equations with the square of the lcm of the denominators of p, q and dividing them by the square of the numerators of p, q we can replace p, q by integer parameters. The common integer factor k must divide the determinant of the matrix of coefficients of p^2, pq, q^2 which is 1, therefore $k = \pm 1$. Substituting these into (6) we obtain

$$p^4 - (2c+4)p^2q^2 + 4q^4 = \pm \frac{4}{d_1^2}. \qquad (10)$$

Further by $p = a - b, q = b$ we obtain

$$a^4 - 4a^3b + (2-2c)a^2b^2 + (4c+4)ab^3 + (1-2c)b^4 = \pm \frac{4}{d_1^2}. \qquad (11)$$

This parametric family of quartic Thue equations was solved by Bo He, B.Jadrijević, A.Togbé [9]. We use their result.

For $d_1 = 1$ the right hand side of (11) is ± 4. By [9] the solutions are $(a,b) = \pm(1,1)$ for all c and additionally $(a,b) = \pm(3,1)$ for $c = 1$. We calculate $p = a - b, q = b$ and x, y, z from (9). We obtain $(x,y,z) = (x_1,y_1,z_1) = \pm(-2c-4, 0, 1)$ for all c and additionally $(x,y,z) = (x_1,y_1,z_1) = \pm(-2,2,1)$ for $c = 1$. The coordinates in the integral basis are $(f_2,f_3,f_4) = (-c-2, 0, 1)$ for all c and $(f_2,f_3,f_4) = (-1,2,1)$ for $c = 1$.

For $d_1 = 2$ the right hand side of (11) is ± 1. We have $(a,b) = \pm(1,0)$ for all c and additionally $(a,b) = \pm(0,1), \pm(2,1)$ for $c = 1$. Hence $(x,y,z) = (2x_1, 2y_1, 2z_1) = \pm(2,0,0)$ for all c and additionally $(x,y,z) = (2x_1,2y_1,2z_1) = \pm(-10,2,2), \pm(-10,-2,2)$ for $c = 1$. The corresponding coordinates in the integral basis are $(f_2,f_3,f_4) = (1,0,0)$ for all c and $(f_2,f_3,f_4) = (-5, \pm 2, 2)$ for $c = 1$.

B. The case of even parameters c

The calculation of the possible α, β, γ for $m = 1, 2, \ldots, 32$ according to (5) leads to the tuples

$$(m,u,v,\alpha,\beta,\gamma) = (4,32,0,32,32,32), (32,64,0,64,64,64) \qquad (12)$$

valid for any c (with $v = 0$) and to 32 tuples of special values of $(m,c,u,v,\alpha,\beta,\gamma)$ (with $v \neq 0$).

B1. The case $v = 0$, arbitrary c

We proceed the same way as in Case A and finally get to the equation

$$a^4 - 4a^3b + (2-2c)a^2b^2 + (4c+4)ab^3 + (1-2c)b^4 = \pm \frac{2^s}{d_1^2} = \eta, \qquad (13)$$

where $s = 5$ or $s = 6$ since $\pm u = 2^5$ or 2^6. We have $m = 4, 32$ and d_1^6 divides the right hand side of the equation. Therefore we have the following cases:

$$(m, s, d_1, \eta) = (4, 5, 1, \pm 32), (4, 5, 2, \pm 8), (4, 5, 4, \pm 2),$$
$$(32, 6, 1, \pm 64), (32, 6, 2, \pm 16), (32, 6, 4, \pm 4), (32, 6, 8, \pm 1).$$

The result of Bo He, B.Jadrijević, A.Togbé [9] gives the primitive solutions of (13) with right hand side $\leq \max\{c/4, 4\}$. We need the solutions with right hand sides $\eta = \pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \pm 32, \pm 64$.

B1.0.

- If $c \geq 256$, then $c/4 \geq 64$, [9] gives the solutions for all right hand sides.
- If $128 \leq c < 256$ then $c/4 \geq 32$, [9] gives the solutions for $\eta = \pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \pm 32$ and we solve the equation with $\eta = \pm 64$.
- If $64 \leq c < 128$ then $c/4 \geq 16$, [9] gives the solutions for $\eta = \pm 1, \pm 2, \pm 4, \pm 8, \pm 16$ and we solve the equation with $\eta = \pm 32, \pm 64$.
- If $32 \leq c < 64$ then $c/4 \geq 8$, [9] gives the solutions for $\eta = \pm 1, \pm 2, \pm 4, \pm 8$ and we solve the equation with $\eta = \pm 16, \pm 32, \pm 64$.
- If $c < 32$ then $c/4 \geq 4$, [9] gives the solutions for $\eta = \pm 1, \pm 2, \pm 4$ and we solve the equation with $\eta = \pm 8, \pm 16, \pm 32, \pm 64$.

Note that the solutions (a, b) of (13) with $\gcd(a, b) = 1$ are called primitive solutions. For solutions with $\gcd(a, b) = g$ we have $g^4 | \eta$ whence $g = 1$ or $g = 2$, the later is only possible for $\eta = \pm 16, \pm 32, \pm 64$. By $\gcd(x_1, y_1, z_1) = 1$ we must have $\gcd(p, q) = \gcd(a, b) = 1$, therefore we are only looking for primitive solutions.

B1.1. The result of Bo He, B.Jadrijević, A.Togbé [9] gives the following solutions in the cases listed in B1.0:

By [9], for all $c \equiv 2 \pmod 4$, $c \neq 2$, equation (13) has primitive solutions only if $\eta = 1, 4$, which means that we consider cases $(m, s, d_1, \eta) = (32, 6, 4, 4), (32, 6, 8, 1)$. Consequently, we have non-primitive solutions (a, b) (those with $\gcd(a, b) = 2$) only if $\eta = 64$ and 16. In that case we have $(m, s, d_1, \eta) = (32, 6, 1, 64), (32, 6, 2, 16)$.

a. For all $c \equiv 2 \pmod 4$, $c \neq 2$ the solutions are $(a,b) = \pm(1,1)$ if $\eta = 4$ and $(a,b) = \pm(1,0)$ if $\eta = 1$. We calculate $p = a-b, q = b$ and (x,y,z) from (9). We obtain $(x_1,y_1,z_1) = \pm(-2c-4,0,1)$, $\pm(1,0,0)$ and

$$(x,y,z) = (4x_1,4y_1,4z_1) = \pm(-8\,(c+2)\,,0,4)$$
$$(x,y,z) = (8x_1,8y_1,8z_1) = \pm(8,0,0)$$

resulting

$$(f_2,f_3,f_4) = (-c-3,0,4) \text{ for } c \equiv 2 \pmod 8$$
$$(f_2,f_3,f_4) = (-c-5,0,4) \text{ for } c \equiv 6 \pmod 8$$
$$(f_2,f_3,f_4) = (1,0,0) \text{ for } c \equiv 2,6 \pmod 8.$$

b. If $c = 8l^2 - 2, (l \in \mathbb{N})$, then the additional solutions are $(a,b) = \pm(4l+1,1)$, $\pm(4l-1,-1)$ if $\eta = 4$ and $(a,b) = \pm(2l+1,2l)$, $\pm(2l-1,2l)$ if $\eta = 1$. We calculate $p = a-b, q = b$ and x,y,z from (9).

For $\eta = 4$ we have $(x_1,y_1,z_1) = \pm(0,\pm 4l,1)$ whence $(x,y,z) = (4x_1,4y_1,4z_1) = \pm(0,\pm 16l,4)$. Therefore
$(f_2,f_3,f_4) = \pm(\pm 4l-3,\pm 8l,4)$.
For $\eta = 1$ we have
$(x_1,y_1,z_1) = \pm(1-64l^4,2l,4l^2)$
whence
$(x,y,z) = (8x_1,8y_1,8z_1) = \pm(8(1-64l^4),\pm 16l,32l^2)$.
This is the element with
$(f_2,f_3,f_4) = (-64l^4-24l^2 \pm 4l+1,\pm 8l,32l^2)$.

Remark. In case when $\eta = 64$ and 16 all solutions are of the form $(a,b) = (2a_0,2b_0)$, where (a_0,b_0) is a primitive solution of (13) for $\eta = 4$ and 1, respectively. Let (x_1',y_1',z_1'), (x_1,y_1,z_1) be triples, given by (9), corresponding to the solution $(2a_0,2b_0)$, (a_0,b_0), respectively. Then $\gcd(x_1',y_1',z_1') = 4$ and $(x_1',y_1',z_1') = (4x_1,4y_1,4z_1)$. Since, for $\eta = 64$ and $\eta = 16$ we have $d_1 = 1$ and $d_1 = 2$, respectively, we obtain

$$\begin{aligned}(x,y,z) &= (x_1',y_1',z_1') = (4x_1,4y_1,4z_1), \quad \text{if} \quad \eta = 64,\\ (x,y,z) &= (2x_1',2y_1',2z_1') = (8x_1,8y_1,8z_1), \quad \text{if} \quad \eta = 16.\end{aligned}$$

Therefore, when $\eta = 64$ and 16 we obtain solutions (x, y, z) and corresponding elements (f_2, f_3, f_4) that we already obtained in cases $\eta = 4$ and 1.

B1.2. For the other cases listed in B1.0 (not covered by [9]) we explicitly solve the Thue equations (13) by using Magma. We obtained solutions only for $\eta = 64$ and 16. For all possible pairs (c, η) with $\eta = 64$ and 16, we find that the solutions of equation (13) are of the form $(a, b) = (2a_0, 2b_0)$, where (a_0, b_0) is a primitive solution of (13) for $\eta = 4$ and $\eta = 1$, respectively. Note, that those primitive solutions (a_0, b_0) we have obtained in case B1.1.a and additional solutions for $c = 6, 30, 70, 126, 198$ all of which are of type $8l^2 - 2$, case B1.1.b. Therefore there are no additional solutions (f_2, f_3, f_4).

B2. The case of $v \neq 0$, given c

By enumerating the possible triples α, β, γ and solving the system of equations (5) we found 32 tuples of special values of $(m, c, u, v, \alpha, \beta, \gamma)$ with $v \neq 0$.

For these 32 cases we followed the method of [7] and calculated the corresponding solutions. The quartic Thue equations were solved by Magma. By investigating these cases we encountered Thue equations with large coefficients and large right hand sides. Here we only give an example which is rather typical.

Example. $(c, m, u, v) = (38, 15, 32, -2)$

$$Q_0(x_1, y_1, z_1) = uQ_2(x_1, y_1, z_1) - vQ_1(x_1, y_1, z_1) = 2(x_1^2 + 144x_1z_1 - 64y_1^2 + 512z_1^2).$$

This quadratic form has the non-trivial solution $x_0 = 8, y_0 = -1, z_0 = 0$. The parametrization

$$x = rx_0 + p = 8r + p, \quad y = ry_0 = -r, \quad z = rz_0 + q = q$$

leads to the representation

$$\begin{aligned} kx_1 &= 8p^2 & & -40992q^2 \\ ky_1 &= p^2 & +144pq & +5124q^2 \\ kz_1 &= & 16pq & +1152q^2 \end{aligned}$$

where p, q are integer parameters and the integer factor k divides 15360. Substituting this representation into

$$Q_2(x_1, y_1, z_1) = \frac{v}{d_1^2}$$

we obtain

$$p^4 + 160p^3q + 1288p^2q^2 - 817356pq^3 - 32690160q^4 = \frac{k^2v}{d_1^2} = \frac{-2k^2}{d_1^2}$$

where d_1 divides $d^6m/n = 2^{13}m = 2^{13} \cdot 15$.

This yields that in this case we have to solve a large number of Thue equations, corresponding to the possible values of k, d_1. The right hand side of the equations goes up to 15.635.001.600. □

From the solutions (p, q) we calculated the corresponding coordinates (x, y, z) finally the coordinates in the integral basis of the elements with index m. In many cases we obtained (x, y, z) corresponding to non-integral elements. We only list here the algebraic integral elements we calculated.
For $c = 6$ elements of index 4 are
$(f_2, f_3, f_4) = (0, 1, 0), (6, -1, -2), (5, 1, -2), (1, -1, 0)$.
For $c = 10$ elements of index 23 are
$(f_2, f_3, f_4) = (10, 0, -3), (-4, 0, 1), (10, 0, -3)$.
For $c = 22$ elements of index 31 are
$(f_2, f_3, f_4) = (20, 0, -3), (-6, 0, 1), (140, 3, -21), (143, -3, -21),$
$(-3, 3, 1), (0, -3, 1), (20, 0, -3)$.

5 Computational aspects

In our calculations we used Maple [2] as well as Magma [1]. Formal calculations and the enumeration of the possible triples α, β, γ were performed by Maple. The resolution of Thue equations was executed in Magma. In case B2 ($v \neq 0$) of the proof of our main Theorem we had 32 cases to consider, all involving a large number of Thue equations (see the Example in our Proof). All together we had to solve thousands of Thue equations, which was performed within 3 hours of CPU time on an average laptop. Magma did not have much difficulty to solve these equations even with large coefficients and large right hand sides.

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