A SYSTEM OF RELATIVE PELLIAN EQUATIONS AND A RELATED FAMILY OF RELATIVE THUE EQUATIONS *

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In this paper we consider the family of systems $(2c+1)U^2 - 2cV^2 = \mu$ and $(c-2)U^2 - cZ^2 = -2\mu$ of relative Pellian equations, where the parameter c and the root of unity μ are integers in the same imaginary quadratic number field $K = \mathbb{Q}\left(\sqrt{-D}\right)$. We show that only for $|c| > 4$ certain values of μ yield solutions of this system, and solve the system completely for $|c| \geq 1544686$. Furthermore we will consider the related relative Thue equation

$$X^4 - 4cX^3Y + (6c+2)X^2Y^2 + 4cXY^3 + Y^4 = \mu$$

and solve it by the method of Tzanakis under the same assumptions.

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1. Introduction and main results

Let $F \in \mathbb{Z}[X,Y]$ be a homogeneous irreducible polynomial of degree $d \geq 3$ and m a nonzero integer. Then the Diophantine equation

$$F(X,Y) = m \tag{1.2}$$

is called a Thue equation in honour of Axel Thue [19], who proved that these equations only have finitely many integral solutions. The proof of this result is based on Thue's theorem on approximations of algebraic numbers. In particular Thue proved that, if $\varepsilon > 0$ and α is some algebraic number of degree $d \geq 2$, then there are only finitely many integers p and $q > 0$ that satisfy

$$\left|\alpha - \frac{p}{q}\right| < q^{-d/2-1-\varepsilon}.$$

Since the proof of this approximation theorem is non-effective, we cannot solve Thue equations by exploiting the proof of Thue. Algorithms for solving a single Thue equation (cf. [6]) have been developed by using Baker's method [1].

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Also various families of Thue equations have been considered dating back to Thue [20], Siegel [17] and many other authors. However, none of the families considered until 1990 had positive discriminant, i.e. all roots of the related polynomial $F(X,1)$ are real. Families of Thue equations with positive discriminant have been first considered by E. Thomas [18]. Since then several such families have been studied up to degree 8 (cf. [13]).

Furthermore, relative Thue equations, i.e. Thue equations with coefficients and solutions that come from the same number field, have been considered (cf. [2]) and also algorithms for solving relative Thue equations have been established by Gaál and Pohst [11]. Moreover, families of relative Thue equations over imaginary quadratic number fields have been considered by Heuberger, Pethő and Tichy [12].

In 1993, Tzanakis [21] considered quartic Thue equations whose corresponding quartic field K is Galois and non-cyclic. Tzanakis showed that a Thue equation of this type corresponds to a system of simultaneous Pellian equations. Various techniques to solve such systems have been developed by Baker and Davenport [3], Mohanty and Ramasamy [14] and Pinch [15] (see also [21]). Using the method of Tzanakis, Dujella and Jadrijević [8] solved the parametrized Thue equation

$$X^4 - 4cX^3Y + (6c+2)X^2Y^2 + 4cXY^3 + Y^4 = 1$$

by reducing it to the system

$$\begin{aligned}(2c+1)U^2 - 2cV^2 &= 1,\\ (c-2)U^2 - cZ^2 &= -2.\end{aligned}$$

of Pellian equations. They solved this system for all rational integers $c \geq 4$ by the method of Baker and Davenport (cf. [3]) combined with the congruence method (cf. [8]) and a result of Bennett [5] on simultaneous approximations of square roots. By a refinement of their proof, Dujella and Jadrijević [9] also solved the family of Thue inequalities

$$|X^4 - 4cX^3Y + (6c+2)X^2Y^2 + 4cXY^3 + Y^4| \leq 6c+4.$$

The aim of this paper is to solve the following system of relative Pellian equations

$$\begin{aligned}(2c+1)U^2 - 2cV^2 &= \mu,\\ (c-2)U^2 - cZ^2 &= -2\mu.\end{aligned} \tag{1.6}$$

In particular, we prove

Theorem 1.1. *Let c be an algebraic integer and μ a root of unity lying in the same imaginary quadratic number field $K = \mathbb{Q}\left(\sqrt{-D}\right)$, where D is a square-free positive integer. If $|c| > 4$ and if system (1.6) of Pellian equations has a solution, then $\mu \in \left\{1, -1, \rho, \rho^2\right\}$ where $\rho = \frac{1}{2}\left(-1+\sqrt{-3}\right)$. Furthermore, if $|c| \geq 1544686$, then the only solutions of (1.6) are $(U,V,Z) = (\pm\varepsilon, \pm\varepsilon, \pm\varepsilon)$ with mixed signs and $\varepsilon = 1, i, \rho^2, \rho$ corresponding to $\mu = 1, -1, \rho, \rho^2$, respectively.*

By the method of Tzanakis we will show in Section 2 that every solution of

$$X^4 - 4cX^3Y + (6c+2)X^2Y^2 + 4cXY^3 + Y^4 = \mu, \tag{1.7}$$

yields a solution of (1.6). Therefore we deduce

Theorem 1.2. *Let the parameter c and the root of unity μ be integers in the same imaginary quadratic number field $K = \mathbb{Q}\left(\sqrt{-D}\right)$, where D is a square-free positive integer, and let $\mathcal{O}_K$ be the ring of algebraic integers in K. If $|c| > 4$ and if equation (1.7) has a solution $(x, y) \in \mathcal{O}_K \times \mathcal{O}_K$, then $\mu \in \{1, -1, \rho, \rho^2\}$ where $\rho = \frac{1}{2}(-1 + \sqrt{-3})$. Furthermore, if $|c| \geq 1544686$ all solutions of (1.7) are given by*

(1) $(x, y) \in \{(0, \pm 1), (\pm 1, 0), (0, \pm i), (\pm i, 0)\} \cap K \times K$ *if* $\mu = 1$.
(2) $(x, y) \in \{(0, \pm \rho), (\pm \rho, 0)\} \cap K \times K$ *if* $\mu = \rho^2$.
(3) $(x, y) \in \{(0, \pm \rho^2), (\pm \rho^2, 0)\} \cap K \times K$ *if* $\mu = \rho$.

Note that in the case of $\mu = -1$ there is no solution of (1.7) if $|c| \geq 1544686$.

The rest of this paper is organized as follows. In the next section we associate solutions of system (1.6) of Pellian equations with solutions of relative Thue equation (1.7). In the following Sections 3, 4 and 5 we will consider the quadratic Diophantine equation $(k-1)X^2 - (k+1)Y^2 = -2\mu$ and prove in Section 5 first results about system (1.6). In Section 6 we apply the "congruence method" to obtain lower bounds for $|U|$. Combining these results with an extension of Bennett's theorem (cf. [5]) proved in Section 7 we can solve system (1.6) for large values of $|c|$. After this, the proof of Theorem 1.2 will follow immediately from the results of Section 2.

2. The method of Tzanakis

In this section we want to show how the family of relative Thue equations (1.7) corresponds to system (1.6) of Pellian equations. We start with the relative Thue equation

$$X^4 - 4cX^3Y + (6c+2)X^2Y^2 + 4cXY^3 + Y^4 = \mu, \tag{2.8}$$

where the parameter c, the root of unity μ, and the solutions x and y are integers in the same imaginary quadratic number field $K = \mathbb{Q}\left(\sqrt{-D}\right)$, with $D > 0$.

By the method of Tzanakis [21] and the paper of Dujella and Jadrijević [8], every solution $(x, y) \in \mathcal{O}_K \times \mathcal{O}_K$ of relative Thue equation (2.8) yields a solution of the system

$$\begin{aligned} (2c+1)U^2 - 2cV^2 &= \mu, \\ (c-2)U^2 - cZ^2 &= -2\mu, \end{aligned} \tag{2.9}$$

where

$$U = x^2 + y^2, \qquad V = x^2 + xy - y^2, \qquad Z = -x^2 + 4xy + y^2.$$

In order to obtain this system one only needs to repeat the computations from [8]. These computations are still valid in $\mathcal{O}_K$.

3. Remarks on relative Pellian equations

In order to get expressions of the form $\sqrt{c}$ uniquely determined we will assume that $\sqrt{c}$ always has positive imaginary part or is a non-negative real number.

Let δ and $\sigma \neq 0$ be integers in the same imaginary quadratic number field $K = \mathbb{Q}\left(\sqrt{-D}\right)$, where D is a square-free positive rational integer and δ is not a perfect square in K. Then we call an equation of the form

$$X^2 - \delta Y^2 = \sigma \tag{3.11}$$

a *relative Pellian equation.* Let $\mathcal{O}_K$ be the ring of algebraic integers of K. If $x, y \in \mathcal{O}_K$ satisfy (3.11) , i.e. $x^2 - \delta y^2 = \sigma$, then we say that $x + y\sqrt{\delta}$ is a *solution* of (3.11). Solutions $x + y\sqrt{\delta}$ of (3.11) in which either x or y are zero are called *improper*. By a solution we will always mean a *proper* solution, i.e. a solution in which both x and y are non-zero. The (proper) solutions of (3.11) always occur in groups of four, namely $\pm x \pm y\sqrt{\delta}$ with mixed signs.

Let us first discuss the case $\sigma = 1$. In this case there exists a solution $x_1 + y_1\sqrt{\delta}$ such that all proper solutions are given by $\pm x_n \pm y_n\sqrt{\delta}$, with mixed signs and $x_n + y_n\sqrt{\delta} = \left(x_1 + y_1\sqrt{\delta}\right)^n$. We call $x_1 + y_1\sqrt{\delta}$ a *fundamental* solution. Obviously there is only one set of solutions that is fundamental.

Let us now consider Pellian equations with $\sigma \neq 1$. If $u + v\sqrt{\delta}$ is a solution of (3.11), $x + y\sqrt{\delta}$ is a solution to the corresponding equation

$$X^2 - \delta Y^2 = 1$$

and

$$\left(x + y\sqrt{\delta}\right)\left(u + v\sqrt{\delta}\right) = s + t\sqrt{\delta},$$

then $\pm s \pm t\sqrt{\delta}$ with mixed signs are solutions of (3.11) too. All solutions of the form $\pm s \pm t\sqrt{\delta}$ coming from some fixed solution $u + v\sqrt{\delta}$ form a *class* C of solutions. A solution $u + v\sqrt{\delta}$ such that the norm $\mathrm{N}(u)$ is the least possible that occurs in this class C is called fundamental. Of course, a fundamental solution $u + v\sqrt{\delta}$ of a given class C is uniquely determined up to $\pm u \pm v\sqrt{\delta}$.

Note that our definition of the class of solutions differs from the usual one. We call the union of a class and its conjugate (in the usual sense) a class of solutions. Since we deal with parametric relative Pellian equations, the unique fundamental solution of the particular class (in the usual sense) depends on the parameter from $\mathcal{O}_K$, and for this reason its uniqueness is difficult to define. Since there is no need to define uniqueness, we omit it.

4. The equation $X^2 - (k^2 - 1)Y^2 = 1$

If we consider system (2.9) we observe that both equations are of the form

$$(k-1)X^2 - (k+1)Y^2 = -2\mu. \tag{4.14}$$

Indeed, let $k = c - 1$ for the second equation and multiply the first equation by -2 and substitute $k = 4c + 1$. Therefore, we consider equations of the form given by (4.14). Multiplying equation (4.14) by $(k - 1)$, excluding the case $k = 1$ and substituting $X_1 = (k - 1)X$ we obtain the Pellian equation

$$X_1^2 - (k^2 - 1)Y^2 = -2\mu(k - 1). \tag{4.15}$$

In the case $k = 1$, one can reduce equation (4.14) to equation $Y^2 = \mu$, i.e. if $c = 0$ or $c = 2$, system (2.9) can be reduced to a single equation $U^2 = \mu$ or to the system $5U^2 - 4V^2 = \mu$, $Z^2 = \mu$, respectively.

If $k^2 - 1$ is a square, the left side of (4.15) splits, which we want to exclude. Let us therefore assume $k^2 - 1 = t^2$, with $t \in \mathcal{O}_K$. Obviously we have $(k - t)(k + t) = 1$ hence $k - t$ and $k + t$ are both units. Checking all (finitely many) cases, we see that the only possibilities are $k = 0, \pm 1$. Furthermore we remark that $k^2 - 1$ can only be a unit if $k = 0$, which is excluded. To see this, note that $k^2 - 1 = (k - 1)(k + 1)$ is a unit if and only if $\varepsilon := k - 1$ and $\varepsilon + 2$ are both units which is equivalent to $\varepsilon = -1$ respectively $k = 0$. In order to find all solutions of equation (4.15) first we have to consider the corresponding Pell equation

$$X_1^2 - (k^2 - 1)Y^2 = 1. \tag{4.16}$$

Therefore we prove

Lemma 4.1. *Let $|k| > 1$. Then $k + \sqrt{k^2 - 1}$ is a fundamental solution of Pell equation* (4.16)*.*

Proof. In order to prove this lemma we use [10, Theorem 3]. Therefore we only have to prove

$$\mathrm{N}(k) > \tfrac{(D+5)^2}{4(D+1)^2} + \tfrac{1}{4} \qquad \text{if } D \equiv 3 \mod 4,$$

$$\mathrm{N}(k) > \tfrac{(D+1)^2}{4D^2} + \tfrac{1}{4} \qquad \text{if } D \equiv 1, 2 \mod 4,$$

where N denotes the norm, D is square-free and $K = \mathbb{Q}\left(\sqrt{-D}\right)$. Since we assume $|k| > 1$, we have $\mathrm{N}(k) \geq 2$. Therefore the inequalities are valid for every $|k| > 1$ and every field $K = \mathbb{Q}\left(\sqrt{-D}\right)$. □

5. The equation $(k - 1)X^2 - (k + 1)Y^2 = -2\mu$

For the rest of this paper we will assume $|k| > 1$. Under this assumption we have

$$k^2 - 1 \notin \left\{\pm\sqrt{-1}, \pm 2, \pm\frac{1}{2}\left(3 - \sqrt{-3}\right), \pm\frac{1}{2}\left(3 + \sqrt{-3}\right), \pm\sqrt{-3}\right\}$$

and we may apply [10, Theorem 11 and Theorem 12]. First we want to find all fundamental solutions of Pellian equation (4.15) with some extra condition.

Lemma 5.1. *Let $|k| > 8$. Then $x + y\sqrt{k^2 - 1} = \varepsilon(\pm(k - 1) \pm \sqrt{k^2 - 1})$ are all fundamental solutions of* (4.15) *that satisfy the additional condition $(k-1)|x$, where*

$\varepsilon = 1, i, \rho^2, \rho$ corresponds to $\mu = 1, -1, \rho, \rho^2$, with $\rho = \frac{1}{2}(-1+\sqrt{-3})$, respectively. In particular there are no fundamental solutions $x+y\sqrt{k^2-1}$ that satisfy the condition $(k-1)|x$ if $\mu = \pm i, -\rho, -\rho^2$.

Proof. Let $x + y\sqrt{k^2-1}$ be a fundamental solution of equation (4.15). From [10, Theorem 11] and Lemma 4.1 we deduce

$$\begin{aligned} 0 <&|x^2| < 2\frac{|k^2-1|}{|k|-1}|k-1|, \\ 0 \leq&|y^2| < 2\left(\frac{1}{|k|-1} + \frac{1}{|k^2-1|}\right)|k-1|. \end{aligned} \tag{5.18}$$

In particular, if we assume $|k| > 8$ we find $0 \leq |y|^2 < 3$, or equivalently $0 \leq |y|^2 \leq 2$. Since (x, y) is a solution of (4.15) we have

$$\frac{x^2}{(k-1)^2} = \frac{y^2(k^2-1)}{(k-1)^2} - \frac{2\mu(k-1)}{(k-1)^2} = y^2 + \frac{2y^2-2\mu}{k-1}.$$

Since $(k-1)|x$ the quantity $\alpha := \frac{2y^2-2\mu}{k-1}$ must be an algebraic integer of K. If $\alpha \neq 0$, then $|2y^2 - 2\mu| \geq |k-1|$ and since $|y|^2 \leq 2$ we find $|k| \leq 7$. Since we excluded the case $|k| \leq 7$ we have $\alpha = 0$. Hence $y^2 = \mu$ and $x^2 = (k-1)^2y^2 = (k-1)^2\mu$. Therefore μ must be a square, i.e. $\mu \in \{1, -1, \rho, \rho^2\}$. Taking square roots yields all possibilities for x and y and moreover all fundamental solutions, with the extra condition $(k-1)|x$. □

From the Lemmas 4.1 and 5.1 together with [10, Theorem 12] we obtain the following proposition:

Proposition 5.2. *Let k, with $|k| > 1$, be an algebraic integer and μ a root of unity lying in the same imaginary quadratic number field $K := \mathbb{Q}(\sqrt{-D})$ with square-free positive integer D. Suppose $|k| > 3$ or k is not an element of the set*

$$\begin{aligned}\mathcal{S} = \big\{&\pm3, \pm2, -2\pm\sqrt{-1}, \pm1\pm\sqrt{-1}, \pm1\pm2\sqrt{-1}, \pm2\sqrt{-1}, \pm\sqrt{-2}, \pm1\pm\sqrt{-2}, \pm\sqrt{-3}, \\ &\pm\tfrac{1}{2}(\pm3\pm\sqrt{-3}), \pm1\pm\sqrt{-3}, \pm2\pm\sqrt{-3}, \pm\sqrt{-5}, \tfrac{1}{2}(\pm1\pm\sqrt{-7}), \tfrac{1}{2}(\pm3\pm\sqrt{-7}), \\ &\tfrac{1}{2}(\pm1\pm\sqrt{-11}), \tfrac{1}{2}(-3\pm\sqrt{-11}), \tfrac{1}{2}(\pm1\pm\sqrt{-15}), \tfrac{1}{2}(\pm1\pm\sqrt{-19})\big\}\end{aligned}$$

with mixed signs. If equation (4.14) has a solution then $\mu \in \{1, -1, \rho, \rho^2\}$, where $\rho = \frac{1}{2}\left(-1+\sqrt{-3}\right)$. Let $\varepsilon = 1, i, \rho^2, \rho$ corresponding to $\mu = 1, -1, \rho, \rho^2$, respectively. Then all solutions are of the form $x = \pm x_m$, $y = \pm y_m$, or $x = \pm x'_m$, $y = \pm y'_m$ with mixed signs, where the sequences $(x_m), (y_m), (x'_m)$ and (y'_m) satisfy the recurrence relations

$$\begin{aligned} x_0 =&\varepsilon, & x_1 =&\varepsilon(2k+1), & x_{m+2} =&2kx_{m+1} - x_m, \quad m \geq 0; \\ y_0 =&\varepsilon, & y_1 =&\varepsilon(2k-1), & y_{m+2} =&2ky_{m+1} - y_m, \quad m \geq 0; \\ x'_0 =&\varepsilon, & x'_1 =&-\varepsilon, & x'_{m+2} =&2kx'_{m+1} - x'_m, \quad m \geq 0; \\ y'_0 =&-\varepsilon, & y'_1 =&\varepsilon, & y'_{m+2} =&2ky'_{m+1} - y'_m, \quad m \geq 0. \end{aligned}$$

Proof. Let $s+t\sqrt{k^2-1}$ be a solution of equation (4.15) from a given class C. Then there exists $n \in \mathbb{N}$ such that

$$s^* + t^*\sqrt{k^2-1} = \left(x_n + y_n\sqrt{k^2-1}\right)\left(x + y\sqrt{k^2-1}\right), \tag{5.19}$$

where $x + y\sqrt{k^2-1}$ is the fundamental solution of the class C, $x_n + y_n\sqrt{k^2-1} = \left(k + \sqrt{k^2-1}\right)^n$, and $s^* + t^*\sqrt{k^2-1}$ is one of the elements from the set $\left\{\pm s \pm t\sqrt{k^2-1}\right\}$. From (5.19) we have $s = x_n x + y_n y\left(k^2-1\right)$ or $s = -x_n x - y_n y\left(k^2-1\right)$. Suppose that $\frac{s}{k-1} \in \mathcal{O}_K$. Then $\frac{x_n x}{k-1} \in \mathcal{O}_K$. Since

$$x_n = k^n + \sum_{j=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} k^{n-2j}\left(k^2-1\right)^j,$$

we have $x\frac{k^n}{k-1} = x\frac{k^n-1}{k-1} + x\frac{1}{k-1} \in \mathcal{O}_K$. This implies $\frac{x}{k-1} \in \mathcal{O}_K$. Therefore, if $s + t\sqrt{k^2-1}$ is a solution of equation (4.15) of a given class C which satisfies the additional condition $(k-1)|s$, then the corresponding fundamental solution $x + y\sqrt{k^2-1}$ of the class C also satisfies the additional condition $(k-1)|x$. Now, the case $|k| > 8$ is a direct consequence of Lemmas 4.1 and 5.1 together with [10, Theorem 12]. All other cases have been computed separately by computer using (5.18) and the fact that $(k-1)|x$. Similarly, one can obtain all fundamentals solutions if $|k| > 1$, but there are too many to write them all down. $\square$

Remark 5.3. In all cases we have

$$\begin{aligned} x_0' &= -y_0, & x_1' &= y_0, & x_{m+1}' &= y_m, & m &\geq 1,\\ y_0' &= x_0, & y_1' &= -x_0, & y_{m+1}' &= -x_m, & m &\geq 1, \end{aligned}$$

hence all solutions of Pellian equation (4.14) are given by $(x,y) = (\pm x_m, \pm y_m)$, $(\pm y_m, \pm x_m)$, with $m \geq 0$.

As a corollary we obtain

Corollary 5.4. *Let c and the root of unity μ be integers in the same imaginary quadratic number field $K = \mathbb{Q}(\sqrt{-D})$, where D is a square-free positive integer, and let $\mathcal{O}_K$ be the ring of algebraic integers in K. If $|c| > 4$ and if system* (2.9) *of Pellian equations has a solution, then $\mu \in \left\{1, -1, \rho, \rho^2\right\}$ with $\rho = \frac{1}{2}\left(-1+\sqrt{-3}\right)$. Furthermore, if $(U,V,Z) \in \mathcal{O}_K^3$ is a solution of* (2.9)*, then there exist nonnegative integers m and n such that*

$$U = \pm v_m = \pm w_n,\ \ U = \pm v_m' = \pm w_n,\ \ U = \pm v_m = \pm w_n' \ \text{ or } \ U = \pm v_m' = \pm w_n',$$

where the sequences (v_m), (w_n), (v'_m) and (w'_n) are given by the following recurrence relations

$$\begin{array}{llll} v_0=\varepsilon, & v_1=\varepsilon(8c+1), & v_{m+2}=(8c+2)v_{m+1}-v_m, & m\geq 0,\\ w_0=\varepsilon, & w_1=\varepsilon(2c-1), & w_{n+2}=(2c-2)w_{n+1}-w_n, & n\geq 0,\\ v'_0=\varepsilon, & v'_1=\varepsilon(8c+3), & v'_{m+2}=(8c+2)v'_{m+1}-v'_m, & m\geq 0,\\ w'_0=\varepsilon, & w'_1=\varepsilon(2c-3), & w'_{n+2}=(2c-2)w'_{n+1}-w'_n, & n\geq 0, \end{array} \tag{5.20}$$

where $\varepsilon=1,i,\rho,\rho^2$ corresponds to $\mu=1,-1,\rho,\rho^2$.

Let us consider the sequences defined by (5.20).

Lemma 5.5. *Let $|c|>2$, then*

$$\begin{array}{ll} (8|c|-3)^m\leq |v_m|\leq (8|c|+3)^m, & (8|c|-3)^m\leq |v'_m|\leq (8|c|+3)^m,\\ (2|c|-3)^n\leq |w_n|\leq (2|c|+3)^n, & (2|c|-3)^n\leq |w'_n|\leq (2|c|+3)^n. \end{array}$$

In particular if $\pm v_m=\pm w_n$, $\pm v'_m=\pm w_n$, $\pm v_m=\pm w'_n$ or $\pm v'_m=\pm w'_n$ then $n>m$ or $n=m=0$.

Proof. First we want to prove that the sequences $(|v_m|)$, $(|w_n|)$, $(|v'_m|)$ and $(|w'_n|)$ are increasing. In all cases this is obviously true for $n,m\in\{0,1\}$. Therefore, let us suppose that $|v_m|\geq |v_{m-1}|$. By the first recurrence in (5.20) we obtain

$$\begin{aligned} |v_{m+1}|=&|(8c-2)v_m-v_{m-1}|\geq (8|c|-2)|v_m|-|v_{m-1}|\\ \geq&(8|c|-3)|v_m|+|v_m|-|v_{m-1}|\geq |v_m|. \end{aligned} \tag{5.21}$$

By induction the sequence $(|v_m|)$ is increasing. If we replace v_m by v'_m we see that the inequality (5.21) is valid for v'_m too, hence $(|v'_m|)$ is also increasing. Assume now that $|w_n|\geq |w_{n-1}|$. Then, by the second recurrence relation in (5.20), we obtain

$$\begin{aligned} |w_{n+1}|=&|(2c-2)w_n-w_{n-1}|\geq (2|c|-2)|w_n|-|w_{n-1}|\\ \geq&(2|c|-3)|w_n|+|w_n|-|w_{n-1}|\geq |w_n|. \end{aligned} \tag{5.22}$$

Hence our claim follows by induction. Replacing w_n by w'_n shows that inequality (5.22) is also valid for w'_n.

For $n,m\in\{0,1\}$ the inequalities from Lemma 5.5 are obviously true. Let us assume that the first inequality is true for v_{m-2} and v_{m-1}. By the first recurrence of (5.20) we have

$$\begin{aligned} |v_m|\leq (8|c|+2)|v_{m-1}|+|v_{m-2}|=&(8|c|+3)|v_{m-1}|-|v_{m-1}|+|v_{m-2}|\leq (8|c|+3)^m;\\ |v_m|\geq (8|c|-2)|v_{m-1}|-|v_{m-2}|=&(8|c|-3)|v_{m-1}|+|v_{m-1}|-|v_{m-2}|\geq (8|c|-3)^m. \end{aligned} \tag{5.23}$$

Let us now suppose that the third inequality from Lemma 5.5 is true for w_{n-2} and w_{n-1}. Then we obtain

$$\begin{aligned} |w_n|\leq (2|c|+2)|w_{n-1}|+|w_{n-2}|=&(2|c|+3)|w_{n-1}|-|w_{n-1}|+|w_{n-2}|\leq (2|c|+3)^n;\\ |w_n|\geq (2|c|-2)|w_{n-1}|-|w_{n-2}|=&(2|c|-3)|w_{n-1}|+|w_{n-1}|-|w_{n-2}|\geq (2|c|-3)^n. \end{aligned} \tag{5.24}$$

Since (5.23) and (5.24) are still valid if we replace v_m by v'_m and w_n by w'_n respectively, we have proved the first statement of the lemma. The relations between n and m follow immediately. □

6. The congruence method

The aim of this section is to find a lower bound for $|U|$ by the so called "congruence method". First we have to define what congruences are in an imaginary quadratic number field.

Definition 6.1. *Let a, b, d be integers in $K = \mathbb{Q}\left(\sqrt{-D}\right)$ and suppose $d \neq 0$. We say that a is congruent b modulo d, if there exists $x \in \mathcal{O}_K$ such that $a - b = dx$. Then we write $a \equiv b \pmod{d}$.*

Lemma 6.2. *Let the sequences (v_m), (w_n), (v'_m) and (w'_n) be defined by (5.20). Using the notations as in Corollary 5.4 for all $m, n \geq 0$ we have*

$$\begin{aligned} v_m &\equiv \varepsilon(4m(m+1)c+1) \pmod{64c^2}, \\ w_n &\equiv \varepsilon(-1)^{n-1}[n(n+1)c-1] \pmod{4c^2}, \\ v'_m &\equiv \varepsilon(2m+1) \pmod{8c}, \\ w'_n &\equiv \varepsilon(-1)^n(2n+1) \pmod{2c}. \end{aligned} \tag{6.25}$$

Proof. We prove all four congruence relations by induction. All four relations are obviously true for $m, n \in \{0, 1\}$.

Assume that the first and the third congruence of (6.25) is valid for $m-2$ and $m-1$. Then we have

$$\begin{aligned} v_m &= (8c+2)v_{m-1} - v_{m-2} \\ &\equiv (8c+2)\varepsilon[4m(m-1)c+1] - \varepsilon[4(m-1)(m-2)c+1] \\ &\equiv \varepsilon c[8 + 8m(m-1) - 4(m-1)(m-2)] + 1 \\ &= \varepsilon(4m(m+1)c+1) \pmod{64c^2}, \end{aligned}$$

and

$$v'_m = (8c+2)v'_{m-1} - v'_{m-2} \equiv 2\varepsilon(2m-1) - \varepsilon(2m-3) = \varepsilon(2m+1) \pmod{8c}.$$

Similarly, if we assume that the second and the last congruence of (6.25) is true for $n-2$ and $n-1$, then we obtain

$$\begin{aligned} w_n &= (2c-2)w_{n-1} - w_{n-2} \\ &\equiv (2c-2)(-1)^n\varepsilon[n(n-1)c-1] - (-1)^{n-1}\varepsilon[(n-1)(n-2)c-1] \\ &\equiv \varepsilon(c(-1)^{n-1}[2 + 2n(n-1) - (n-1)(n-2)] + (-1)^n) \\ &= (-1)^{n-1}\varepsilon[n(n+1)c-1] \pmod{4c^2}, \end{aligned}$$

and

$$\begin{aligned} w'_n =&(2c-2)w'_{n-1} - w'_{n-2} \equiv -2\varepsilon(-1)^{n-1}(2n-1) - (-1)^{n-2}\varepsilon(2n-3) \\ =&\varepsilon(-1)^n(2n+1) \pmod{2c}. \end{aligned}$$

□

Now we will discuss all four cases listed in Corollary 5.4.

Let us start with $v_m = \pm w_n$. Suppose that m and n are positive integers such that $v_m = \pm w_n$. Then, of course, $v_m \equiv \pm w_n \pmod{4c^2}$. Because of Lemma 6.2, we have $\varepsilon \equiv \pm(-1)^n\varepsilon \pmod{2c}$. In the case of $-\varepsilon \equiv \varepsilon \pmod{2c}$ we have $\varepsilon \equiv 0 \pmod{c}$, which implies $|c| = 1$. Therefore, if $|c| > 1$, then n is even in the case that the sign $+$ holds and n is odd in the other case.

Assume now $n(n+1) < \frac{4}{5}|c|$ and $|c| > 1$. Since $m < n$ (cf. Lemma 5.5) we also have $m(m+1) < \frac{4}{5}|c|$. Furthermore, Lemma 6.2 together with the previous paragraph implies

$$\varepsilon(4m(m+1)c+1) \equiv \varepsilon(1-n(n+1)c) \pmod{4c^2}$$

in both cases. Therefore,

$$\varepsilon 2m(m+1) \equiv -\varepsilon\frac{n(n+1)}{2} \pmod{2c}. \tag{6.27}$$

Consider

$$A = \varepsilon\left(2m(m+1) + \frac{n(n+1)}{2}\right) \in \mathcal{O}_K.$$

We have $0 < |A| < 2|c|$. By (6.27), $A \equiv 0 \pmod{2c}$, which implies $|A| \geq 2|c|$, a contradiction. Hence $n(n+1) \geq \frac{4}{5}|c|$, i.e. $n > \sqrt{0.8|c|} - 0.5$.

Now let us consider the case $v'_m = \pm w_n$, where m and n are positive integers. We find $v'_m \equiv \pm w_n \pmod{2c}$ and by Lemma 6.2, we have $\varepsilon(2m+1) \equiv \pm(-1)^n\varepsilon \pmod{2c}$ or equivalently $\varepsilon(m + 1/2 \pm 1/2) \equiv 0 \pmod{c}$, which implies $n > m \geq |c| - 1$.

Suppose that m and n are positive integers such that $v_m = \pm w'_n$. This yields the congruence $\varepsilon \equiv \pm(-1)^n\varepsilon(2n+1) \pmod{2c}$ and similarly we obtain $\varepsilon(n+1/2 \pm 1/2) \equiv 0 \pmod{c}$ which implies $n \geq |c| - 1$.

At last, if we assume m and n are positive integers such that $v'_m = \pm w'_n$, then by Lemma 6.2 we obtain $\varepsilon(2m+1) \equiv \pm\varepsilon(-1)^n(2n+1) \pmod{2c}$. Since $n > m$ we have $2n+1 \pm (2m+1) = 0$ or $2n+1 \pm (2m+1) \geq 2|c|$. The first case yields $n = m = 0$ (cf. Lemma 5.5) which is excluded. The second case yields $2n+1 > |c|$ or equivalently $n > |c|/2 - 1/2$.

Therefore we have proved

Proposition 6.3. *Let $|c| \geq 4$. If $v_m = \pm w_n$, $v'_m = \pm w_n$, $v_m = \pm w'_n$ or $v'_m = \pm w'_n$, then $n > \sqrt{0.8|c|} - 0.5$ or $n = m = 0$.*

7. A theorem of Bennett

In 1998 Bennett [5] proved a theorem on simultaneous approximations by rationals to the square roots of rationals near 1. He used this theorem to bound the number of solutions of simultaneous Pell equations. Since we work with imaginary quadratic integers we have to prove an analog of Bennett's theorem in the case of imaginary quadratic number fields.

Theorem 7.1. *Let* $\vartheta_i = \sqrt{1+\frac{a_i}{T}}$ *for* $1 \leq i \leq m$, *with* a_i *pairwise distinct imaginary quadratic integers in* $K := \mathbb{Q}(\sqrt{-D})$ *with* $0 < D \in \mathbb{Z}$ *for* $i = 0, \dots, m$ *and let* T *be an algebraic integer of* K. *Furthermore, let* $M := \max |a_i|$, $|T| > M$ *and* $a_0 = 0$ *and*

$$l = c_m \frac{(m+1)^{m+1}}{m^m} \cdot \frac{|T|}{|T|-M}, \quad L = |T|^m \cdot \frac{(m+1)^{m+1}}{4m^m \prod_{0 \leq i < j \leq m} |a_i - a_j|^2} \cdot \left(\frac{|T|-M}{|T|}\right)^m,$$

$$p = \sqrt{\frac{2|T|+3M}{2|T|-2M}}, \qquad P = |T| \cdot 2^{m+3} \frac{\prod_{0 \leq i < j \leq m} |a_i - a_j|^2}{\min_{i \neq j} |a_i - a_j|^{m+1}} \cdot \frac{2|T|+3M}{2|T|},$$

where $c_m = \frac{3\Gamma(m-1/2)}{4\sqrt{\pi}\Gamma(m+2)}$, *such that* $L > 1$, *then*

$$\max\left(\left|\vartheta_1 - \frac{p_1}{q}\right|, \dots, \left|\vartheta_m - \frac{p_m}{q}\right|\right) > c\,|q|^{-\lambda}$$

for all algebraic integers $p_1, \dots, p_m, q \in K$, *where*

$$\lambda = 1 + \frac{\log P}{\log L} \quad \text{and}$$
$$c^{-1} = 2mpP(\max\{1, 2l\})^{\lambda - 1}.$$

Remark 7.2. The condition $L > 1$ is always fulfilled if we assume

$$|T| > (4M)^{m+1}.$$

Indeed, $|a_j - a_i| \leq 2M$ and therefore

$$\begin{aligned} L \geq & \frac{(m+1)^{m+1}(|T|-M)^m}{4m^m(2M)^{m(m+1)}} \\ > & \frac{(m+1)^{m+1}\left((3M)^{m+1}\right)^m}{4m^m(2M)^{m(m+1)}} \\ > & \frac{(m+1)^{m+1}}{4m^m} \cdot \left(\frac{3}{2}\right)^{m(m+1)} > 1. \end{aligned}$$

In order to prove Theorem 7.1 we will prove first two lemmas from which the theorem immediately follows. The first lemma is due to Chudnovsky [7] who proved an ineffective version valid for rational integers only. We follow a proof of Rickert (cf. [16]), who proved a version of Lemma 7.3 in the integer case.

Lemma 7.3. *Let $\vartheta_1, \ldots, \vartheta_m$ be arbitrary complex numbers and let $\vartheta_0 = 1$. Suppose there exist positive real numbers l, p, L, P, with $L > 1$ such that for every positive integer k we can find algebraic integers $p_{ijk} \in \mathbb{Q}(\sqrt{-D})$, with $0 < D \in \mathbb{Z}$ and*

$$\begin{aligned} \det(p_{ijk}) &\neq 0 & & (0 \leq i, j \leq m), \\ |p_{ijk}| &< pP^k & & (0 \leq i, j \leq m) \ \text{ and} \\ \left|\overbrace{\sum_{j=0}^{m} p_{ijk}\vartheta_j}^{:=l_{ik}}\right| &< lL^{-k} & & (0 \leq i \leq m). \end{aligned}$$

Then we may conclude

$$\max\left(\left|\vartheta_1 - \frac{p_1}{q}\right|, \ldots, \left|\vartheta_m - \frac{p_m}{q}\right|\right) > c|q|^{-\lambda}$$

for all algebraic integers $p_1, \ldots, p_m, q \in \mathbb{Q}(\sqrt{-D})$, where

$$\begin{aligned} \lambda &= 1 + \frac{\log P}{\log L} \quad \text{and} \\ c^{-1} &= 2mpP(\max\{1, 2l\}^{\lambda-1}). \end{aligned}$$

Proof. Let

$$\delta := \max\left(\left|\vartheta_1 - \frac{p_1}{q}\right|, \ldots, \left|\vartheta_m - \frac{p_m}{q}\right|\right).$$

Put $p_0 = q$ and

$$\eta_{ik} = \sum_{j=0}^{m} p_{ijk}p_j.$$

Note that

$$q\sum_{j=0}^{m} p_{ijk}\vartheta_j - \eta_{ik} = q\sum_{j=0}^{m} p_{ijk}(\vartheta_j - p_j/q) \tag{7.30}$$

has absolute value at most $mpP^k\delta|q|$, hence, by the left side of (7.30), we have

$$|\eta_{ik}| \leq mpP^k\delta|q| + lL^{-k}|q|. \tag{7.31}$$

The assumption $\det(p_{ijk}) \neq 0$ shows that for each k the numbers η_{ik} with $0 \leq i \leq m$ are not all zero. Fix i with $0 \leq i \leq m$ and $\eta_{ik} \neq 0$. Since η_{ik} is a non-zero imaginary quadratic integer we have $|\eta_{ik}| \geq 1$. Let $C := \max\{1, 2l\}$, then there exists a positive integer k with $L^k \geq C|q|$ and we deduce $lL^{-k}|q| \leq \frac{1}{2}$. Combining this with (7.31) we obtain that $\delta \neq 0$.

Let us find now a lower bound for δ. Note that the integer

$$k = 1 + \left\lfloor \frac{\log C|q|}{\log L} \right\rfloor$$

certainly satisfies $L^k \geq C|q|$. Using the bounds for $|\eta_{ik}|$ and $lL^{-k}|q|$ in (7.31) we have

$$mpP^k\delta|q| \geq \frac{1}{2}$$

or equivalently

$$mp\delta|q| \geq \frac{1}{2}P^{-k}. \tag{7.32}$$

Moreover, we have

$$P^{-k} \geq P^{-1-\log C|q|/\log L} = P^{-1}(C|q|)^{-\log P/\log L} = P^{-1}(C|q|)^{1-\lambda}. \tag{7.33}$$

The two inequalities (7.32) and (7.33) yield together

$$\delta \geq \frac{1}{2mpPC^{\lambda-1}}|q|^{-\lambda} = c|q|^{-\lambda}.$$

□

Lemma 7.4. *Let $\vartheta_i = \sqrt{1+\frac{a_i}{T}}$, with $i = 1, \dots, m$ and a_i pairwise distinct algebraic integers lying in $K = \mathbb{Q}(\sqrt{-D})$ and let T be an algebraic integer of K. Furthermore, let $M := \max|a_i|$, $|T| > M$, $a_0 = 0$ and $c_m = \frac{3\Gamma(m-1/2)}{4\sqrt{\pi}\Gamma(m+2)}$, then we may take*

$$l = c_m\frac{(m+1)^{m+1}}{m^m}\cdot\frac{|T|}{|T|-M}, \quad L = |T|^m\cdot\frac{(m+1)^{m+1}}{4m^m\prod_{0\leq i<j\leq m}|a_i-a_j|^2}\cdot\left(\frac{|\mathbf{T}|-\mathbf{M}}{|\mathbf{T}|}\right)^m,$$

$$p = \sqrt{\frac{2|T|+3M}{2|T|-2M}}, \qquad P = |T|\cdot 2^{m+3}\frac{\prod_{0\leq i<j\leq m}|a_i-a_j|^2}{\min_{i\neq j}|a_i-a_j|^{m+1}}\cdot\frac{2|T|+3M}{2|T|},$$

in order to apply Lemma 7.3, provided $L > 1$.

Proof. Following Rickert's paper [16] we derive our approximations from the contour integral

$$I_i(x;k,\gamma) := \frac{1}{2\pi i}\oint_\gamma \frac{(1+zx)^{k+1/2}}{(z-a_i)A(z)^k}dz,$$

where $A(z) = (z-a_0)\cdots(z-a_m)$ and γ is a closed, counter-clockwise contour enclosing the poles of the integrand. From a lemma of Rickert [16, Lemma 3.3] we obtain

$$I_i(x) = \sum_{j=0}^m p_{ij}(x)(1+a_jx)^{1/2},$$

with

$$p_{ij}(x) = \sum\binom{k+1/2}{h_j}(1+a_jx)^{k-h_j}x^{h_j}\prod_j^*\binom{-k_{il}}{h_l}(a_j-a_l)^{-k_{il}-h_l},$$

where $k_{ij} = k+\delta_{ij}$, the sum is taken over all non-negative $h_0, \dots, h_m$ with sum $k_{ij}-1$ and the product is taken over all $l = 0$ to $l = m$ omitting the index j. From this representation of p_{ij} we have by Rickert [16, Lemma 4.3] respectively Bennett [5]

$$\left(4x\prod_{0\leq i<j\leq m}(a_i-a_j)^2\right)^k p_{ij}\left(\frac{1}{x}\right)\in\mathbb{Z}[x,a_0,\dots,a_m].$$

Furthermore, we have by Cauchy's theorem

$$p_{ij}(x)=\frac{1}{(1+a_jx)^{1/2}}\cdot\frac{1}{2\pi i}\oint_{\Gamma_j}\frac{(1+zx)^{k+1/2}}{(z-a_i)A(z)^k}dz,$$

where Γ_j is a closed curve enclosing only the pole at a_j. Therefore, we choose for our approximations

$$p_{ijk}:=\overbrace{\left(4T\prod_{0\leq i<j\leq m}(a_i-a_j)^2\right)^k}^{:=C_k} p_{ij}\left(\frac{1}{T}\right)=\frac{C_k}{(1+a_j/T)^{1/2}}\cdot\frac{1}{2\pi i}\oint_{\Gamma_j}\frac{(1+z/T)^{k+1/2}}{(z-a_i)A(z)^k}dz$$

and

$$l_{ik}:=C_kI_i\left(\frac{1}{T};k,\gamma\right).$$

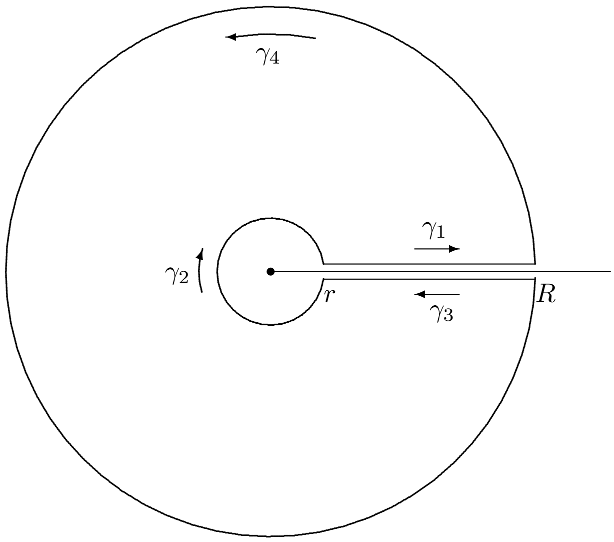
Let us first analyze $I_i\left(\frac{1}{T}\right)$ using the ideas of Bennett (cf. [4]). Therefore let γ be the curve such that after the transformation $z\to -xT-T$ we obtain the curve γ' which encloses all poles of the integrand while avoiding a branch cut along the non-negative real axis (see Figure 1 and cf. [4]).

The transformation $z\to -xT-T$ yields

$$\begin{aligned}I_i\left(\frac{1}{T}\right)=&\frac{1}{2\pi i}\oint_{\gamma}\frac{(1+z/T)^{k+1/2}}{(z-a_i)A(z)^k}dz\\ =&\frac{(-1)^{k+1+1/2}T}{2\pi i}\oint_{\gamma'}\frac{\exp((k+\frac{1}{2})\log(x))}{(-Tx-T-a_i)A(-Tx-T)^k}dx\\ =&\frac{(-1)^{mk+1/2}}{2\pi iT^{(m+1)k}}\oint_{\gamma'}\frac{\exp((k+\frac{1}{2})\log(x))}{(x+1+\frac{a_i}{T})B(x)^k}dx,\end{aligned}$$

where $B(x)=(x+1+\frac{a_0}{T})\cdots(x+1+\frac{a_m}{T})$. Since the absolut value of the last integrand is bounded for small $|x|$ and is of order $O\left(|x|^{-mk-\frac{1}{2}}\right)$ for large $|x|$, the integrals along γ_2 and γ_4 become negligible as $r\to 0$ and $R\to\infty$. Therefore we

Figure 1. The curve γ after the transformation $z \to -xT - T$.



conclude, letting $r \to 0$ and $R \to \infty$

$$\begin{aligned}
\left|I_i\left(\frac{1}{T}\right)\right| =&\frac{1}{2\pi|T|^{(m+1)k}}\left|(1-e^{\pi i})\int_0^\infty \frac{x^{k+1/2}}{(x+1+a_i/T)B(x)^k}dx\right|\\
\leq&\frac{\left(1-\frac{M}{|T|}\right)^{-mk-1}}{\pi|T|^{(m+1)k}}\int_0^\infty \frac{x^{k+1/2}}{(x+1)^{(m+1)k+1}}dx\\
=&\frac{\left(1-\frac{M}{|T|}\right)^{-mk-1}}{\pi|T|^{(m+1)k}}\frac{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(mk-\frac{1}{2}\right)}{\Gamma((m+1)k+1)}\\
\leq&\frac{3\Gamma(m-\frac{1}{2})}{4\sqrt{\pi}\Gamma(m+2)}\frac{(m+1)^{m+1}}{m^m}\cdot\frac{|T|}{|T|-M}\times\\
&\left(|T|^m\cdot\frac{(m+1)^{m+1}}{m^m}\cdot\left(\frac{|T|-M}{|T|}\right)^m\right)^{-k}
\end{aligned}$$

The last inequlity holds, since (see Lemma 7.6)

$$\frac{\Gamma(k+3/2)\Gamma(mk-1/2)}{\Gamma((m+1)k+1)} \leq \left(\frac{m^m}{(m+1)^{m+1}}\right)^{k-1}\frac{3\sqrt{\pi}\Gamma(m-1/2)}{4\Gamma(m+2)},$$

which is proved at the end of this section.

Let us estimate now

$$\tilde{p}_{ij} = \left(1 + \frac{a_j}{T}\right)^{1/2} p_{ij}\left(\frac{1}{T}\right) = \frac{1}{2\pi i}\oint_{\Gamma_j} \frac{\left(1+\frac{z}{T}\right)^{k+1/2}}{(z-a_i)A(z)^k}dz,$$

where Γ_j is the contour defined by

$$|z - a_j| := \min_{i \neq j}\left(\frac{|a_i - a_j|}{2}\right).$$

Then we have

$$\begin{aligned}
|\tilde{p}_{ij}| &= \left|\frac{1}{2\pi i}\oint_{\Gamma_j} \frac{\left(1+\frac{z}{T}\right)^{k+1/2}}{(z-a_i)A(z)^k}dz\right| \\
&\leq \left|\frac{1}{2\pi i}\oint_{\Gamma_j} \frac{1}{z-a_j}dz \cdot \max_{z\in\Gamma_j}\left|\frac{\left(1+\frac{z}{T}\right)^{k+1/2}(z-a_j)}{(z-a_i)A(z)^k}\right|\right| \\
&= \left|\max_{z\in\Gamma_j}\left|\frac{\left(1+\frac{z}{T}\right)^{k+1/2}(z-a_j)}{(z-a_i)A(z)^k}\right|\right| \\
&\leq \left(\frac{2^{m+1}}{\min_{i\neq j}|a_i-a_j|^{m+1}}\right)^k \max_{z\in\Gamma_j}\left|\left(1+\frac{z}{T}\right)^{k+1/2}\right|,
\end{aligned}$$

since $|z - a_i| \geq \left|\frac{a_i - a_j}{2}\right|$. Furthermore, because $|z| \leq \frac{3M}{2}$ we obtain

$$|\tilde{p}_{ij}| \leq \left(\frac{2|T|+3M}{2|T|}\right)^{1/2}\left(\frac{2^{m+1}}{\min_{i\neq j}|a_i-a_j|^{m+1}} \cdot \frac{2|T|+3M}{2|T|}\right)^k.$$

The lemma follows now from putting together the various bounds. □

Corollary 7.5. *If we assume* $m = 2$, $a_0 = 0, a_1 = -4, a_2 = 1$ *and* $|T| \geq 3.04 \cdot 10^6$ *we may choose*

$$l = \frac{106875}{253333} < 0.42188, \qquad L = |T|^2\frac{15595158960027}{3696640000000000} > |T|^2 \cdot 0.004218,$$

$$p = \sqrt{\frac{1520003}{1519998}} < 1.0001, \qquad P = |T|\frac{6080012}{475} < |T| \cdot 12800.03,$$

and furthermore

$$\max_{1\leq i\leq 2}\left|\sqrt{1+\frac{a_i}{T}} - \frac{p_i}{q}\right| > \frac{1}{51200.63|T|}q^{-1-(\log|T|+9.4573)/(2\log|T|-5.469)}.$$

In particular, we have $2 - \lambda > 0$.

In order to complete the proof of Theorem 7.1 we have to prove

Lemma 7.6.

$$\frac{\Gamma(k+3/2)\Gamma(mk-1/2)}{\Gamma((m+1)k+1)} \leq \left(\frac{m^m}{(m+1)^{m+1}}\right)^{k-1}\frac{3\sqrt{\pi}\Gamma(m-1/2)}{4\Gamma(m+2)}$$

Proof. Proof by Induction in k. The case $k = 1$ is trivial, so we have to show

$$\frac{\frac{\Gamma(k+3/2)\Gamma(mk-1/2)}{\Gamma((m+1)k+1)}}{\frac{\Gamma(k+1/2)\Gamma(m(k-1)-1/2)}{\Gamma((m+1)(k-1)+1)}} \leq \frac{m^m}{(m+1)^{m+1}}.$$

By Γ's functional equation ($x\Gamma(x) = \Gamma(x+1)$) we have

$$\frac{(k+1/2)\prod_{l=0}^{m-1}(m(k-1)-1/2+l)}{\prod_{l=0}^{m}((m+1)(k-1)+1+l)} \leq \frac{m^m}{(m+1)^{m+1}}$$

or equivalently

$$\frac{k+1/2}{k(m+1)} \prod_{l=0}^{m-1} \frac{m(k-1)-1/2+l}{(m+1)(k-1)+1+l} \leq \frac{m^m}{(m+1)^{m+1}}.$$

Since the function $f(x) = \frac{a+x}{b+x} = 1 - \frac{b-a}{b+x}$ is monotony increasing if $b > a$, the factor $\frac{m(k-1)-1/2+l}{(m+1)(k-1)+1+l}$ takes its maximum for $l = m-1$. Note that $m(k-1)-1/2 \leq (m+1)(k-1)+1$. Therefore it suffices to prove

$$\frac{k+1/2}{k(m+1)} \left(\frac{m(k-1)-1/2+m-1}{(m+1)(k-1)+1+m-1}\right)^m \leq \frac{m^m}{(m+1)^{m+1}}$$

or equivalently

$$\left(1+\frac{1}{2k}\right)\left(\frac{mk-3/2}{(m+1)k-1}\right)^m \leq \left(\frac{m}{m+1}\right)^m$$

or

$$1+\frac{1}{2k} \leq \left(\frac{m^2k+mk-m}{m^2k+mk-3m/2-3/2}\right)^m = \left(1+\frac{m+3}{(m+1)(2km-3)}\right)^m.$$

Using Bernoulli's inequality we obtain

$$\begin{aligned}\left(1+\frac{m+3}{(m+1)(2km-3)}\right)^m &\geq 1+\frac{m(m+3)}{(m+1)(2km-3)} \\ &\geq 1+\frac{m(m+1)}{(m+1)(2km)} = 1+\frac{1}{2k}.\end{aligned}$$

□

8. Proof of our main results

Let $(U, V, Z) \in \mathcal{O}_K^3$ be a solution of system (2.9) of Pellian equations. Let us define $\vartheta_1^{(1)} = \pm\sqrt{\frac{2c+1}{2c}}$, $\vartheta_1^{(2)} = -\vartheta_1^{(1)}$ and $\vartheta_2^{(1)} = \pm\sqrt{\frac{c-2}{c}}$, $\vartheta_2^{(2)} = -\vartheta_2^{(1)}$ respectively, where we choose the signs such that

$$\begin{aligned}\left|V-\vartheta_1^{(1)}U\right| &\leq \left|V-\vartheta_1^{(2)}U\right|, \\ \left|Z-\vartheta_2^{(1)}U\right| &\leq \left|Z-\vartheta_2^{(2)}U\right|.\end{aligned}$$

Similarly, as in the case of Thue equations, solutions to (2.9) induce good approximations to $\vartheta_1^{(1)}$ and $\vartheta_2^{(1)}$. More precisely, we have

Lemma 8.1. *If $|c| > 2$, then all solutions $(U, V, Z) \in \mathcal{O}_K^3$ of the system of Pellian equations* (2.9) *satisfy*

$$\begin{aligned} \left| \vartheta_1^{(1)} - \frac{V}{U} \right| \leq & \frac{1}{\sqrt{3}|c|} \cdot |U|^{-2}, \\ \left| \vartheta_2^{(1)} - \frac{Z}{U} \right| \leq & \frac{2}{\sqrt{|c|(|c|-2)}} \cdot |U|^{-2}. \end{aligned}$$

Proof. We have

$$\begin{aligned} \left| V - \vartheta_1^{(2)} U \right| \geq & \frac{1}{2} \left(\left| V - \vartheta_1^{(1)} U \right| + \left| V - \vartheta_1^{(2)} U \right| \right) \geq \frac{1}{2} \left| U \left(\vartheta_1^{(1)} - \vartheta_1^{(2)} \right) \right| \\ \geq & |U| \sqrt{\left| \frac{2c+1}{2c} \right|} \geq |U| \sqrt{\frac{2|c|-1}{2|c|}} \geq |U| \frac{\sqrt{3}}{2}, \end{aligned}$$

and similarly

$$\begin{aligned} \left| Z - \vartheta_2^{(2)} U \right| \geq & \frac{1}{2} \left(\left| Z - \vartheta_2^{(1)} U \right| + \left| Z - \vartheta_2^{(2)} U \right| \right) \geq \frac{1}{2} \left| U \left(\vartheta_2^{(1)} - \vartheta_2^{(2)} \right) \right| \\ \geq & |U| \sqrt{\left| \frac{c-2}{c} \right|} \geq |U| \sqrt{\frac{|c|-2}{|c|}}. \end{aligned}$$

This implies

$$\begin{aligned} \left| \vartheta_1^{(1)} - \frac{V}{U} \right| = & \left| \frac{2c+1}{2c} - \frac{V^2}{U^2} \right| \cdot \left| \vartheta_1^{(2)} - \frac{V}{U} \right|^{-1} \\ \leq & \frac{1}{|2cU^2|} \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}|c|} \cdot |U|^{-2}, \end{aligned}$$

and

$$\begin{aligned} \left| \vartheta_2^{(1)} - \frac{Z}{U} \right| = & \left| \frac{c-2}{c} - \frac{Z^2}{U^2} \right| \cdot \left| \vartheta_2^{(2)} - \frac{Z}{U} \right|^{-1} \\ \leq & \frac{2}{|cU^2|} \sqrt{\frac{|c|}{|c|-2}} = \frac{2}{\sqrt{|c|(|c|-2)}} \cdot |U|^{-2}. \end{aligned}$$

□

Now let us apply Corollary 7.5 to $\vartheta_1^{(1)}$, $\vartheta_2^{(1)}$ and $T = 2c$ where $|c| \geq 1.52 \cdot 10^6$. Together with Lemma 8.1 we obtain the following inequalities

$$\frac{1}{51200.63 \cdot 2|c|} \cdot |U|^{-\lambda} < \max \left(\left| \vartheta_1^{(1)} - \frac{V}{U} \right|, \left| \vartheta_2^{(1)} - \frac{V}{U} \right| \right) \leq \frac{2}{\sqrt{|c|(|c|-2)}} \cdot |U|^{-2}. \quad (8.42)$$

By Corollary 7.5 we have $2 - \lambda > 0$, hence inequality (8.42) cannot hold for large $|U|$. However, taking logarithms yields

$$\log |U| < \frac{\log \left(\frac{2}{\sqrt{|c|(|c|-2)}} \cdot 51200.63 \cdot 2|c| \right)}{2 - \lambda} < \frac{12.23}{2 - \lambda}. \quad (8.43)$$

Furthermore,

$$\frac{1}{2-\lambda} < \frac{1}{1-\frac{\log 12800.03\cdot 2|c|}{\log 0.004218\cdot 4|c|^2}} < \frac{\log\left(1.6874\cdot 10^{-2}\cdot |c|^2\right)}{\log\left(6.5913\cdot 10^{-7}\cdot |c|\right)}.$$

On the other hand, Lemma 5.5 together with Proposition 6.3 implies that if $(m,n)\neq(0,0)$, then

$$|U| > (2|c|-3)^{\sqrt{0.8|c|}-0.5}.$$

Therefore,

$$\log|U| > \left(\sqrt{0.8|c|}-0.5\right)\log(2|c|-3). \tag{8.46}$$

Combining inequalities (8.43) and (8.46) we obtain

$$\sqrt{0.8|c|}-0.5 < \left(12.23\cdot\frac{\log\left(1.6874\cdot 10^{-2}\cdot |c|^2\right)}{\log\left(6.5913\cdot 10^{-7}\cdot |c|\right)}\right)\cdot\frac{1}{\log(2|c|-3)} \tag{8.47}$$

which does not hold for $|c|\geq 1544686$.

Therefore $U\in\{\pm v_0,\pm v_0',\pm w_0,\pm w_0'\}=\{\pm\varepsilon\}$ where $\varepsilon=1,i,\rho^2,\rho$ corresponding to $\mu=1,-1,\rho,\rho^2$, respectively, and Theorem 1.1 follows immediately. Now let us prove Theorem 1.2.

Proof of Theorem 1.2. The first statement follows immediately from the remarks in Section 2 and Corollary 5.4.

Let (x,y) be a solution of (1.7) and let $|c|\geq 1544686$. We know from Theorem 1.1 that

$$U=x^2+y^2=\pm\varepsilon,\qquad V=x^2+xy-y^2=\pm\varepsilon,\qquad Z=-x^2+4xy+y^2=\pm\varepsilon,$$

where $\varepsilon\in\{1,i,\rho^2,\rho\}$. Adding V and Z yields $5xy=0,\pm 2\varepsilon$. Because $5\nmid 2$ we have $5xy=0$, hence either x or y is zero, i.e. $x^4=\mu$ and $y=0$ or $x=0$ and $y^4=\mu$. □

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