On elements with index of the form 2^a3^b in a parametric family of biquadratic fields

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Abstract

In this paper we give some results about primitive integral elements α in the family of bicyclic biquadratic fields $L_c = \mathbb{Q}\left(\sqrt{(c-2)\,c}, \sqrt{(c+4)\,c}\right)$ which have index of the form $\mu(\alpha) = 2^a3^b$ and coprime coordinates in given integral bases. Precisely, we show that if $c \geq 11$ and α is an element with index $\mu(\alpha) = 2^a3^b \leq c+1$, then α is an element with minimal index $\mu(\alpha) = \mu(L_c) = 12$. We also show that for every integer $C_0 \geq 3$ we can find effectively computable constants $M_0(C_0)$ and $N_0(C_0)$ such that if $c \leq C_0$, than there are no elements α with index of the form $\mu(\alpha) = 2^a3^b$, where $a > M(C_0)$ or $b > N(C_0)$.

1 Introduction

Let α be a primitive integral element of an algebraic number field K of degree n with ring of integers $\mathcal{O}_K$. Then index of α is defined as index of subgroup $\mathbb{Z}[\alpha]^+$ in group $\mathcal{O}_K^+$

$$\mu(\alpha) = \left(\mathcal{O}_K^+ : \mathbb{Z}[\alpha]^+\right),$$

where $\mathcal{O}_K^+$ and $\mathbb{Z}[\alpha]^+$ denote the additive groups of corresponding rings. The minimal index $\mu(K)$ of the field K we define as the minimum of the indices of all primitive integers in the field K. The field index $m(K)$ is the greatest common divisor of indices also taken for all primitive integers of K.

Let $\{1, \omega_2, ..., \omega_n\}$ be an arbitrary integral basis of K. Then discriminant of corresponding linear form $L(\underline{X}) = X_1 + \omega_2X_2 + ... + \omega_nX_n$ can be rewritten as

$$D_{K/\mathbb{Q}}(L(\underline{X})) = (I(X_2, ..., X_n))^2 D_K,$$

where D_K is discriminant of the field K and $I(X_2, ..., X_n)$ is a homogeneous polynomial in $n-1$ variables of degree $n(n-1)/2$ with rational integer coefficients. The polynomial $I(X_2, ..., X_n)$ is called the index form associated to the

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integral basis $\{1, \omega_2, ..., \omega_n\}$. If the primitive integral element α is represented in that integral basis as $\alpha = x_1 + x_2\omega_2 + ... + x_n\omega_n$, $x_1, x_2, ...x_n \in \mathbb{Z}$, then the index of α is just $\mu(\alpha) = |I(x_2, ..., x_n)|$. Hence, the problem of determining elements of given index $\mu \in \mathbb{N}$ can be reduced to the solving index form equations

$$I(x_2, ..., x_n) = \pm\mu \text{ in } x_2, ..., x_n \in \mathbb{Z}.$$

Bicyclic biquadratic fields are quartic fields of the type $\mathbb{Q}(\sqrt{m}, \sqrt{n})$, where m, n are distinct square-free rational integers. These fields were considered several authors. M. N. Gras, and F. Tanoe [9] have found necessary and sufficient conditions for biquadratic fields being monogenic. I. Gaál, A. Pethő and M. Pohst [7] gave an algorithm for determining the minimal index and all elements with minimal index in the totally real case using the integral basis described by K. S. Williams [14]. G. Nyul [12] classified all monogene totally complex biquadratic fields and gave explicitly all generators of power integral bases in them. In [10] and [11] the author has determined the minimal index and all elements with minimal index for three infinite families of totally real bicyclic biquadratic fields. Further, I. Gaál and G. Nyul [8] provided an efficient algorithm for determining elements of index divisible by fixed primes in biquadratic number fields.

In [11] the author proved following theorem.

Theorem 1 *Let $c \geq 3$ be an integer such that $c \equiv 1$ or $3 \pmod 6$ and c, $c-2$, $c+4$ are square-free integers. Then*

$$L_c = \mathbb{Q}\left(\sqrt{c(c-2)}, \sqrt{c(c+4)}\right) \tag{1}$$

is a totally real bicyclic biquadratic field and

i) *its field index is $m(L_c) = 1$ for all c;*

ii) *the minimal index of L_c is $\mu(L_c) = 12$ if $c \geq 7$ and $\mu(L_c) = 1$ if $c = 3$;*

iii) *all integral elements with minimal index are given by*

$$x_1 + x_2\sqrt{(c-2)(c+4)} + x_3\frac{\sqrt{(c-2)(c+4)} + \sqrt{(c-2)c}}{2} + x_4\frac{1+\sqrt{c(c+4)}}{2}, \tag{2}$$

where $x_1 \in \mathbb{Z}$, $(x_2, x_3, x_4) = \pm(0,1,1), \pm(0,1,-1), \pm(1,-1,-1), \pm(1,-1,1)$ if $c \geq 7$ and $(x_2, x_3, x_4) = \pm(-1,1,0), \pm(0,1,0)$ if $c = 3$.

Since the minimal index of the field (1) is of the form 2^a3^b, we wonder if there exist primitive integral elements α with the index $\mu(\alpha)$ of this form except those with the minimal index. It suffices to observe elements α of the form (2) with $\gcd(x_2, x_3, x_4) = 1$ since the index form $I(x_2, x_3, x_4)$ is a homogeneous polynomial of degree 6. For a (partial) answer on this question we need some additional conditions: an upper bound for the index $\mu(\alpha)$ or an upper bound for the parameter c.

The main results of the present paper are given in the following theorems:

Theorem 2 *Let $c \geq 3$ be an integer such that $c \equiv 1$ or $3 \pmod 6$ and c, $c-2$, $c+4$ are square-free integers. If α is a primitive integral element of the field (1) given by (2), where $x_1, x_2, x_3, x_4 \in \mathbb{Z}$ with $\gcd(x_2, x_3, x_4) = 1$ and index of α is of the form $\mu(\alpha) = 2^a 3^b$, where $a \geq 0, b \geq 0$ are integers, then the following holds:*

- ***i)*** *If $c \geq 11$ and $\mu(\alpha) \leq c+1$, then $\mu(\alpha) = 12$. Furthermore, if $c < 11$ and $\mu(\alpha) \leq 12$, then $\mu(\alpha) = 12$ if $c \neq 3$ and $\mu(\alpha) = 1$ or $\mu(\alpha) = 12$ if $c = 3$.*
- ***ii)*** *All elements α with $\mu(\alpha) = 12$ are given by $x_1 \in \mathbb{Z}$, $\pm(x_2, x_3, x_4) = (0, 1, 1)$, $(0, 1, -1)$, $(1, -1, -1)$, $(1, -1, 1)$ except when $c = 3$, in which case we have further solutions $x_1 \in \mathbb{Z}$, $\pm(x_2, x_3, x_4) = (5, -29, 11)$, $(5, -29, -11)$, $(24, 29, -11)$, $(24, -29, -11)$. If $c = 3$, then all elements α with $\mu(\alpha) = 1$ are given by $x_1 \in \mathbb{Z}$, $\pm(x_2, x_3, x_4) = (-1, 1, 0)$, $(0, 1, 0)$.*

Theorem 3 *For every integer $C_0 \geq 3$ we can find effectively computable constants $M_0(C_0)$ and $N_0(C_0)$ such that if $c \leq C_0$, then there are no primitive integral elements α of the field (1) given by (2) with $\gcd(x_2, x_3, x_4) = 1$ and with index of the form $\mu(\alpha) = 2^a 3^b$ where $a > M_0(C_0)$ or $b > N_0(C_0)$.*

Directly from Theorem 2 and Theorem 3 we obtain:

Corollary 1 *For a given parameter c, let $\alpha := \alpha(c)$ denote a corresponding primitive integral element of the field (1) given by (2) with $\gcd(x_2, x_3, x_4) = 1$. Then the following holds:*

- ***i)*** *Let $a \geq 0$ and $b \geq 0$ be arbitrary but fixed integers such that $2^a 3^b > 12$. If there exist a parameter c and an element $\alpha(c)$ with an index $\mu(\alpha(c)) = 2^a 3^b$, then $c \leq 2^a 3^b - 2$.*
- ***ii)*** *Let $c > 3$ be arbitrary but fixed integer. If there exist an element $\alpha(c)$ with index of the form $\mu(\alpha(c)) = 2^a 3^b$, then either $\mu(\alpha(c)) = 12$ or we can find effectively computable constants $M_0(c)$ and $N_0(c)$ such that $c + 2 \leq \mu(\alpha(c)) \leq 2^{M_0(c)} 3^{N_0(c)}$.*

Remark 1 *For the particular value of c there is an efficient algorithm for determining all elements with an index of the form $2^a 3^b$ (see Section 4) based on a more general algorithm given by I. Gaál and G. Nyul [12].*

2 Preliminaries

Note that the field (1) is totally real bicyclic biquadratic field under the assumption c, $c-2$, $c+4$ are positive square-free, pairwise relatively prime integers. Since we use a method of I. Gaál, A. Pethő and M. Pohst [7], we have to observe the congruence behavior of c, $c-2$, $c+4$ modulo 4. Hence, if $c, c-2, c+4$ are positive square-free integers, then $c \geq 3$ and $c \equiv 1$ or $3 \pmod 4$. Note that c, $c-2$, $c+4$ are pairwise relatively prime integers if and only if $c \equiv 1$ or $3 \pmod 6$.

Therefore, we observe cases when $c \geq 3$, $c \equiv 1, 3, 7, 9 \pmod{12}$ and c, $c-2$, $c+4$ are square-free integers. Furthermore, in [11, Section 4] it was shown, by using the result from [5], that there are infinitely many integers c with the above properties which again implies that there are infinitely many totally real bicyclic biquadratic fields of the form (1). Also, in [11, Section 4], by using a method of I. Gaál, A. Pethő and M. Pohst [7], we showed that finding all elements with given index μ is equivalent to finding all solvable systems of the form

$$(c-2)\,U^2 - cV^2 = \pm F_1 \tag{3}$$

$$(c-2)\,Z^2 - (c+4)\,V^2 = \pm F_2 \tag{4}$$

$$cZ^2 - (c+4)\,U^2 = \pm 4F_3, \tag{5}$$

where $F_1 F_2 F_3 = \mu$. Then all integral elements α with index equal to μ are given by (2) where

$$U = 2x_2 + x_3,\ V = x_4,\ Z = x_3, \tag{6}$$

and (U, V, Z) is passing through all solutions of all solvable systems of the form (3), (4) and (5) with $F_1 F_2 F_3 = \mu$. Furthermore, since the equations (3), (4) and (5) are not independent, the relation

$$c\,(\pm F_2) - (c+4)\,(\pm F_1) = (c-2)\,(\pm 4F_3) \tag{7}$$

holds. Therefore, if we want to find all integral elements α of the form (2) with $\gcd(x_2, x_3, x_4) = 1$ and with index $\mu(\alpha) = 2^a 3^b$, then we have to find all solvable systems of the form (3), (4) and (5) with $F_1 F_2 F_3 = 2^a 3^b$ and all solutions (U, V, Z) of these systems which are a form of (6), where $\gcd(x_2, x_3, x_4) = 1$.

We note here that we have two different approaches to this problem, depending on whether we have given an upper bound for the index $\mu(\alpha)$ (as in Theorem 2), or an upper bound for the parameter c (as in Theorem 3). If we have an upper bound for the index $\mu(\alpha)$, then we first consider the system (3) and (5). Namely, we use the theory of continued fractions to determine all possible small values of the right hand side of (3) and (5) so that the system of these two equations has solutions. After that, from equation (7) by direct testing, we find all possible triples $(\pm F_1, \pm F_2, \pm F_3)$ such that $F_1 F_2 F_3$ is of the form $2^a 3^b$. If we have an upper bound on the parameter c, then we first consider the equation (7). Since, in our case F_i, $i = 1, 2, 3$ are of the form $F_i = 2^{\alpha_i} 3^{\beta_i}$, we obtain a S-unit equation over $\mathbb{Z}$. Because we have an upper bound C_0 on the parameter c, we are able to find upper bounds for the exponents a and b using $p-$adic estimates and those upper bounds depend only on C_0. Since our estimates providing large upper bounds for the exponents, we can diminish the upper bounds using reduction procedure. Unfortunately, reduction procedure can be used only for particular values of the parameter c, so reduced upper bounds can not be expressed as a function of C_0.

3 Additional condition: upper bound for the index

We recall that if we want to find all primitive integral elements α of the field (1) with index $\mu(\alpha) = 2^a 3^b$, we have to find all solvable systems of the form (3), (4) and (5), where $F_1 F_2 F_3 = 2^a 3^b$.

Suppose that (U, V, Z) is an integer solution of the system (3), (4) and (5), where $c \geq 3$, $c \equiv 1, 3, 7, 9 \pmod{12}$ and c, $c-2$, $c+4$ are square-free integers. If one of the integers U, V, Z is equal to zero, then (3), (4) and (5) imply that other two integers are not equal to zero. Further, if $U = 0$, then equation (3) implies that c divides F_1 which again implies $c = 3$ since F_1 is of the form $2^\alpha 3^\beta$ and $c \geq 3$ is an odd square-free positive integer. Similarly, we find if $V = 0$, then equation (4) implies $c = 3$. Furthermore, we obtain that there is no c which satisfies the equation (4) if $Z = 0$. Therefore, if $c > 3$, then it is sufficient to observe only solutions (U, V, Z) in positive integers.

Let $c \geq 3$ and let $\mu(\alpha) \leq K$, where K is a positive integer. Further, let (U, V, Z) be a solution in positive integers of the system of Pellian equations (3) and (5). Since $\mu(\alpha) = F_1 F_2 F_3 \leq K$, then $F_1 \leq K$ and $F_3 \leq K$. Now, from (3) and (5), we obtain

$$\left|\sqrt{\frac{c}{c-2}} - \frac{U}{V}\right| = \left|\frac{c}{c-2} - \frac{U^2}{V^2}\right| \cdot \left|\sqrt{\frac{c}{c-2}} + \frac{U}{V}\right|^{-1} < \frac{F_1}{(c-2)V^2} \cdot \sqrt{\frac{c-2}{c}} \leq \frac{K}{\sqrt{c(c-2)}V^2} \tag{8}$$

and

$$\left|\sqrt{\frac{c+4}{c}} - \frac{Z}{U}\right| = \left|\frac{c+4}{c} - \frac{Z^2}{U^2}\right| \cdot \left|\sqrt{\frac{c+4}{c}} + \frac{Z}{U}\right|^{-1} < \frac{4F_3}{cU^2} \cdot \sqrt{\frac{c}{c+4}} \leq \frac{4K}{\sqrt{c(c+4)}U^2}. \tag{9}$$

Note that for $c > 3$ is enough to assume $K \geq 12$, since in that case, by Theorem 1, minimal index is equal to 12.

3.1 Case $c > 3$

Let $c > 3$. Additionally, suppose $12 \leq K \leq c + 1$. Note, that this condition implies $c \geq 11$, and since we have $c \equiv 1, 3, 7, 9 \pmod{12}$, then $c \geq 13$. Under these conditions, from (8) and (9), we obtain that all solutions (U, V, Z) in positive integers of the system of Pellian equations (3) and (5), satisfying

$$\left|\sqrt{\frac{c}{c-2}} - \frac{U}{V}\right| < \frac{K}{\sqrt{c(c-2)}V^2} \leq \frac{c+1}{\sqrt{c(c-2)}V^2} < \frac{2}{V^2} \tag{10}$$

and

$$\left|\sqrt{\frac{c+4}{c}}-\frac{Z}{U}\right|<\frac{4K}{\sqrt{c\left(c+4\right)}U^{2}}\leq\frac{4\left(c+1\right)}{\sqrt{c\left(c+4\right)}U^{2}}<\frac{4}{U^{2}}. \tag{11}$$

Similarly as in [11, Section 4.1], we will use theory of continued fractions to determine all possible values of $\pm F_1$ and $\pm 4F_3$ such that equations (3) and (5) have solutions in relatively prime integers. Precisely, since the inequalities (10) and (11) are satisfied, we can apply Theorem (Worley [15], Dujella [3]) and [4, Lemma 1] (see also [11, Theorem 3 and Lemma 1]).

We find that under above conditions, i.e. if $F_1 \leq c+1$, where $c \geq 13$ and if equation (3) has solutions in relatively prime integers U and V, then

$$\pm F_1 \in S_1\left(c\right)=\left\{-2,-c,c-2\right\}.$$

Since F_1 is of the form $F_1 = 2^{\alpha}3^{\beta}$, where $\alpha \geq 0, \beta \geq 0$ are integers and since $c \equiv 1,3,7,9 \,(\mathrm{mod}\, 12)$, then the only possibility is $\pm F_1 = -2$.

Similarly, if $F_3 \leq c+1$, where $c \geq 13$ and equation (5) has solutions in relatively prime integers U and Z, then

$$\pm 4F_3 \in S_3\left(c\right)=\left\{-4,-1,4c,4c-9,-2c-9,2c-1,-c-4,3c-4,-3c-16\right\}, \text{ if } c>19.$$

Additionally, we have

$$\begin{aligned}
&\pm 4F_3 \in S_3\left(c\right)\cup S_3'\left(c\right) \quad \text{if } c=19,\\
&\pm 4F_3 \in S_3\left(c\right)\cup S_3'\left(c\right)\cup S_3''\left(c\right) \quad \text{if } c=15,\\
&\pm 4F_3 \in S_3\left(c\right)\cup S_3'\left(c\right)\cup S_3''\left(c\right)\cup S_3'''\left(c\right) \quad \text{if } c=13,
\end{aligned}$$

where

$$\begin{aligned}
S_3'\left(c\right)&=\left\{16c-225,5c-16\right\},\\
S_3''\left(c\right)&=\left\{12c-121,14c-169,15c-196,13c-144\right\},\\
S_3'''\left(c\right)&=\left\{6c-25,8c-49,10c-81,7c-36,9c-64,11c-100\right\}.
\end{aligned}$$

Since F_3 is of the form $F_3 = 2^{\gamma}3^{\delta}$, where $\gamma \geq 0, \delta \geq 0$ are integers and since $c \equiv 1,3,7,9 \,(\mathrm{mod}\, 12)$, then the only possibility is $\pm 4F_3 = -4$ for all $c \geq 13$.

Now, suppose that (U,V,Z) is a solution of the system of Pellian equations (3) and (5) in positive integers. Let $\gcd(U,V) = d$ and $\gcd(U,Z) = g$. If $F_1 = 2^{\alpha}3^{\beta} \leq c+1$ and $F_3 = 2^{\gamma}3^{\delta} \leq c+1$, where $c \geq 13$, then $\pm F_1 = -2d^2$ and $\pm 4F_3 = -4g^2$ which implies

$$d \leq \sqrt{\frac{c+1}{2}} < \frac{c}{2} \text{ and } g \leq \sqrt{c+1} < \frac{c}{2}.$$

Let $U = dU_1 = gU_2$, $V = dV_1$ and $Z = gZ_2$. Then $\gcd(U_1,V_1) = 1$, $\gcd(U_2,Z_2) = 1$ and following equations are hold

$$\begin{aligned}
\left(c-2\right)U_1^2 - cV_1^2 &= -2\\
cZ_2^2 - \left(c+4\right)U_2^2 &= -4.
\end{aligned}$$

By [11, Lemma 3], all such U_1 are given recurrently in the following way

$$u_0 = 1, \quad u_1 = 2c-1, \quad u_{m+2} = (2c-2)\, u_{m+1} - u_m, \quad m \geq 0, \tag{12}$$

and all such U_2 are given recurrently by

$$v_0 = 1, \quad v_1 = c+1, \quad v_{n+2} = (c+2)\, v_{n+1} - v_n, \quad n \geq 0. \tag{13}$$

Since $U = dU_1 = gU_2$, then there exist nonnegative integers m and n such that $U = du_m = gv_n$, where u_m and v_n are defined by (12) and (13), respectively. By [11, Lemma 4], for all $m, n \geq 0$, we have

$$u_m \equiv (-1)^{m-1} (m(m+1)c - 1) \pmod{4c^2}, \quad v_n \equiv \frac{n(n+1)}{2} c + 1 \pmod{c^2}.$$

Therefore, if $du_m = gv_n$, then $du_m \equiv gv_n \pmod{c^2}$ which implies $(-1)^m d \equiv g \pmod{c}$. Since $0 < d < \frac{c}{2}$ and $0 < g < \frac{c}{2}$, we have $d = g$, i.e. $U_1 = U_2$. Thus, we obtain a system of simultaneous Pellian equations

$$(c-2)\, U_1^2 - cV_1^2 = -2,$$
$$cZ_2^2 - (c+4)\, U_1^2 = -4.$$

In [11, Theorem 4] we find that for $c \geq 7$ only solutions to this system are $(U_1, V_1, Z_2) = (\pm 1, \pm 1, \pm 1)$. Therefore, all solutions to the corresponding system of Pellian equations (3) and (5) (with $\pm F_1 = -2d^2$, $\pm 4F_3 = -4d^2$ and $d \leq \sqrt{\frac{c+1}{2}}$) are of the form $(U, V, Z) = (\pm d, \pm d, \pm d)$. If $\gcd(x_2, x_3, x_4) = 1$, then (6) implies $\gcd(U, V, Z) = 1$ or 2. Therefore, we have $d = 1$ or 2.

i) If $d = 1$, then $\pm F_1 = -2$, $\pm 4F_3 = -4$, and from (7) we obtain $\pm F_2 = -6$, which implies $\mu(\alpha) = F_1 F_2 F_3 = 2 \cdot 6 \cdot 1 = 12$. Therefore, α has the minimal index, i.e. $\mu(\alpha) = \mu(L_c) = 12$.

ii) If $d = 2$, then from (6), we obtain

$$x_4 = \pm 2, \ 2x_2 + x_3 = \pm 2, \ x_3 = \pm 2,$$

which implies $\pm (x_2, x_3, x_4) = (0, 2, 2), (0, 2, -2), (2, -2, -2), (2, -2, 2)$, a contradiction with $\gcd(x_2, x_3, x_4) = 1$.

When $3 < c < 11$ (which implies $c = 7$), we take $K = 12$. Since, in this case, the minimal index of the field L_c is equal to 12, then $\mu(\alpha) \leq 12$ implies $\mu(\alpha) = \mu(L_c) = 12$.

For each $c > 3$, by Theorem 1, all elements α with minimal index $\mu(\alpha) = \mu(L_c) = 12$ are given by $\pm (x_2, x_3, x_4) = (0, 1, 1), (0, 1, -1), (1, -1, -1), (1, -1, 1)$.

To complete the proof of Theorem 2 it remains to consider the case $c = 3$ and $K = 12$.

3.2 Case $c = 3$ and $K = 12$

Let $c = 3$ and $\mu(\alpha) = F_1F_2F_3 \leq K = 12$. Let (U, V, Z) be a solution in positive integers of the system (3) and (5) with $c = 3$. Since $\mu(\alpha) = F_1F_2F_3 \leq 12$, then $F_1 \leq 12$ and $F_3 \leq 12$. Hence, from (8) and (9), we obtain

$$\left|\sqrt{3} - \frac{U}{V}\right| < \frac{F_1}{\sqrt{3}V^2} \leq \frac{12}{\sqrt{3}V^2} < \frac{7}{V^2} \tag{14}$$

and

$$\left|\sqrt{\frac{7}{3}} - \frac{Z}{U}\right| < \frac{4F_3}{3U^2} \cdot \sqrt{\frac{3}{7}} \leq \frac{4 \cdot 12}{\sqrt{21}U^2} < \frac{11}{U^2}, \tag{15}$$

respectively. If $\gcd(U, V) = 1$ and $F_1 \leq 12$, then from (14) by Theorem (Worley [15], Dujella [3]) and [4, Lemma 1], we obtain

$$\pm F_1 \in S_1' = \{1, -2, -3, 6, -11\}.$$

Knowing that, in our case, F_1 is of the form $F_1 = 2^\alpha 3^\beta$, then the only possibilities are $\pm F_1 = 1, -2, -3, 6$. Similarly, if $\gcd(U, Z) = 1$ and $F_3 \leq 12$ then from (15), we obtain

$$\pm 4F_3 \in S_3 = \{-1, 3, -4, 5, -7, 12, -15, 17, 20, 21, -25, -28, 35, -37, 41, -43, 47\}.$$

Since we have $F_3 = 2^\gamma 3^\delta \leq 12$, then the only possibilities are $\pm 4F_3 = -4, 12$, i.e. $\pm F_3 = -1, 3$.

Additionally, for $c = 3$, equation (7) has form

$$3(\pm F_2) - 7(\pm F_1) = \pm 4F_3, \tag{16}$$

and since $F_1F_2F_3 \leq 12$, we obtain that there are only two possibilities:

i) $(\pm F_1, \pm F_2, \pm F_3) = (1, 1, -1)$ which implies $\mu(\alpha) = 1$. Therefore, $\mu(\alpha)$ is equal to minimal index $\mu(L_3) = 1$. Now, by Theorem 1, all such α are given by $\pm(x_2, x_3, x_4) = (-1, 1, 0), (0, 1, 0)$.

ii) $(\pm F_1, \pm F_2, \pm F_3) = (-2, -6, -1)$ which implies $\mu(\alpha) = 12$. The corresponding system is

$$U^2 - 3V^2 = -2 \tag{17}$$

$$3Z^2 - 7U^2 = -4. \tag{18}$$

Similarly as in [11, Section 4.2], we find that the only solutions to the system (17) and (18) are $(U, V, Z) = (\pm 1, \pm 1, \pm 1)$ and $(U, V, Z) = (\pm 19, \pm 11, \pm 29)$. Since integers U, V, Z are of the form given in (6), where $\gcd(x_2, x_3, x_4) = 1$, all α with index $\mu(\alpha) = 12$ are given by $\pm(x_2, x_3, x_4) = (0, 1, 1)$, $(0, 1, -1)$, $(1, -1, -1)$, $(1, -1, 1)$, $(5, -29, 11)$, $(5, -29, -11)$, $(24, 29, -11)$, $(24, -29, -11)$.

Note that the above results we obtain by assuming (U, V, Z) is a solution in positive integers to the system (3), (4) and (5) with $c = 3$. It remains to observe the cases when (U, V, Z) is solution in nonnegative integers with $U = 0$ or $V = 0$.

If $c = 3$ and $V = 0$, then (3) and (4) imply $U^2 = \pm F_1, \quad Z^2 = \pm F_2$, where $U \neq 0$ and $Z \neq 0$. Therefore, we have $F_1 F_2 = U^2 Z^2 \leq 12$, which implies $(U, V, Z) = (1, 0, 1), (1, 0, 2), (1, 0, 3), (2, 0, 1), (3, 0, 1)$. Since $\pm F_1 = U^2$, $\pm F_2 = Z^2$ and $F_1 F_2 F_3 = 2^a 3^b \leq 12$, from equation (16), we obtain that the only possibility is $(\pm F_1, \pm F_2, \pm F_3) = (1, 1, -1)$ and this triple we have already obtained.

If $U = 0$, then (3) and (5) imply $-3V^2 = \pm F_1, \quad 3Z^2 = \pm 4F_3$, where $V \neq 0$ and $Z \neq 0$. Therefore, we have $F_1 F_3 = \frac{9}{4} V^2 Z^2 \leq 12$ and Z is an even integer. This implies $(U, V, Z) = (0, 1, 2)$. Since $\pm F_1 = -3V^2 = -3$, $\pm 4F_3 = 3Z^2 = 12$, from equation (16), we find $\pm F_2 = -3$. Therefore, we obtain a triple $(\pm F_1, \pm F_2, \pm F_3) = (-3, -3, 3)$ which does not satisfy the condition $F_1 F_2 F_3 \leq 12$. Therefore, we finished the proof of Theorem 2.

4 Additional condition: upper bound for the parameter c

In this section we show that if have an upper bound on the parameter c, then we can find an upper bound for the index. We will follow the method of I. Gaal and G. Nyul given in [8]. We briefly sketch the main steps of our procedure. We start with equation (7). Since, in our case, unknowns F_i in (7), are of the form $F_i = 2^{\alpha_i} 3^{\beta_i}$, $i = 1, 2, 3$, we obtain a S-unit equation over $\mathbb{Z}$. In order to find all elements α with index $\mu(\alpha) = F_1 F_2 F_3 = 2^a 3^b$, we have to find all primitive solution of equation (7) and all possibilities for common factor $2^A 3^B$ of F_1, F_2, F_3 (see Section 4.1). To find all possibilities for the exponents in the common factor $2^A 3^B$, we need to determine how rational primes 2 and 3 split in three distinct quadratic subfields of the quartic field L_c. We show that the exponents A and B attain only very small values (see Section 4.2). If have an upper bound C_0 on the parameter c, then we are able to find an upper bound for the exponents in primitive solutions using $p-$adic linear form estimates and that upper bound depends only on C_0 (see Section 4.3). Since our estimates giving large upper bounds for the exponents, we can diminish those upper bounds using reduction procedure (see Section 4.4). Unfortunately, reduced upper bounds can not be expressed as a function of C_0 since reduction procedure can be used only for particular value of the parameter c.

4.1 S-unit equation

Let α be a primitive integral element of the form (2) with $\gcd(x_2, x_3, x_4) = 1$ and let index of α be $\mu(\alpha) = 2^a 3^b$ where $a \geq 0$ and $b \geq 0$ are arbitrary but fixed integers. We have shown that $\mu(\alpha)$ is of the form $\mu(\alpha) = F_1 F_2 F_3$, where

F_1, F_2, F_3 satisfy relation (7). Since $\mu(\alpha) = 2^a 3^b$ implies

$$F_i = 2^{\alpha_i} 3^{\beta_i},\ 0 \le \alpha_i \le a,\ 0 \le \beta_i \le b,\ i = 1, 2, 3,$$

then (F_1, F_2, F_3) is solution of the S-unit equation (7) over $\mathbb{Z}$.

We will find all primitive solutions (f_1, f_2, f_3) of (7) in positive integers (those with $\gcd(f_1, f_2, f_3) = 1$). Then all solutions of (7) are of the form $F_i = f_i \cdot 2^A 3^B$, $i = 1, 2, 3$, where $2^A 3^B = \gcd(F_1, F_2, F_3) = P$. Set

$$f_i = 2^{d_i} 3^{e_i},\quad i = 1, 2, 3.$$

Then equation $c(\pm f_2) - (c+4)(\pm f_1) = (c-2)(\pm 4 f_3)$ can be rewritten in the form

$$\mp (c+4)\, 2^{d_1} 3^{e_1} \pm c 2^{d_2} 3^{e_2} = \pm (c-2)\, 2^{d_3+2} 3^{e_3}. \tag{19}$$

Note that, since $c \ge 3$, $c \equiv 1, 3, 7, 9 \pmod{12}$ and $c, c-2, c+4$ are square-free integers, we have $ord_2(c+4) = ord_2(c) = ord_2(c-2) = ord_3(c+4) = ord_3(c-2) = 0$ and $ord_3(c) = k$ where $c = 3^k c_1$, $3^k \| c$ and $k = 0$ or 1. Now, if the equation (19) we simplify with possible common factors 2 and 3, we obtain equation

$$\mp (c+4)\, 2^{d_1'} 3^{e_1'} \pm c_1 2^{d_2'} 3^{e_2'} = \pm (c-2)\, 2^{d_3'} 3^{e_3'}, \tag{20}$$

where at most one of the d_1', d_2', d_3' and at most one of the e_1', e_2', e_2' is positive. After determined $d_1', d_2', d_3', e_1', e_2', e_3'$, values of $f_i = 2^{d_i} 3^{e_i}$, $i = 1, 2, 3$, we can obtain using the following:

- If $(d_1', d_2', d_3') = (d_1', 0, 0)$, then $(d_1, d_2, d_3) = (d_1' + 2, 2, 0)$;
- If $(d_1', d_2', d_3') = (0, d_2', 0)$, then $(d_1, d_2, d_3) = (2, d_2' + 2, 0)$;
- If $(d_1', d_2', d_3') = (0, 0, 1)$, then $(d_1, d_2, d_3) = (1, 1, 0)$;
- If $(d_1', d_2', d_3') = (0, 0, d_3')$, $d_3' \ge 2$, then $(d_1, d_2, d_3) = (0, 0, d_3' - 2)$;
- If $c \equiv 1, 7 \pmod{12}$, then $(e_1, e_2, e_3) = (e_1', e_2', e_3')$;
- If $c \equiv 3, 9 \pmod{12}$ and
  - $(e_1', e_2', e_3') = (e_1', 0, 0)$, then $(e_1, e_2, e_3) = (e_1' + 1, 0, 1)$;
  - $(e_1', e_2', e_3') = (0, 0, e_3')$, then $(e_1, e_2, e_3) = (1, 0, e_3' + 1)$;
  - $(e_1', e_2', e_3') = (0, e_2', 0)$, $e_2' \ge 1$, then $(0, e_2, 0) = (0, e_2' - 1, 0)$.

It is easy to see that exponents in (20) cannot all be equal to zero. Furthermore, if $c > 3$, than we have: if there exist i such that $d_i' \ne 0$, then there exist j such that $e_j' \ne 0$, and vice versa. Also, $i \ne j$ must hold. If $c = 3$, then we have the following exceptions: $(d_1', d_2', d_3', e_1', e_2', e_3') = (0, 0, 3, 0, 0, 0)$, $(0, 3, 0, 0, 0, 0)$, $(0, 0, 1, 0, 0, 1)$, $(0, 1, 0, 0, 1, 0)$. Therefore, from now on, we will assume that exactly one d_i' is positive and exactly one e_j' is positive, where $i \ne j$.

4.2 $\gcd(F_1, F_2, F_3)$ calculations

In order to find an upper bound for the exponents in $P = \gcd(F_1, F_2, F_3) = 2^A 3^B$ we will follow of a method of I. Gaal and G. Nyul [8, Section 6]. First, we need to determine how rational primes 2 and 3 splits in three distinct quadratic subfields of the quartic field $L_c = \mathbb{Q}(\sqrt{m}, \sqrt{n})$, namely in the fields $M_1 = \mathbb{Q}(\sqrt{n})$, $M_2 = \mathbb{Q}(\sqrt{m})$, $M_3 = \mathbb{Q}(\sqrt{m_1 n_1})$, where $m_1 = c+4$, $n_1 = c$, $m = (c+4)(c-2)$, $n = c(c-2)$. Using for example [1, p. 245], we find the factorization of the principal ideals $\langle 2\rangle$ and $\langle 3\rangle$ into prime ideals of rings of integers $\mathcal{O}_{M_i}$, $i = 1, 2, 3$ as follows: In the ring $\mathcal{O}_{M_1}$, we have

$$\begin{aligned}
\langle 2\rangle &= \langle 2, 1+\sqrt{n}\rangle^2 = \mathcal{P}_1^2 \\
\langle 3\rangle &= \langle 3, \sqrt{n}\rangle^2 = \mathcal{P}_2^2, \text{ if } c \equiv 3, 9 \pmod{12} \\
\langle 3\rangle &= \mathcal{P}_3, \text{ if } c \equiv 1, 7 \pmod{12}.
\end{aligned}$$

In the ring $\mathcal{O}_{M_2}$, we obtain

$$\begin{aligned}
\langle 2\rangle &= \langle 2, 1+\sqrt{m}\rangle^2 = \mathcal{P}_4^2, \\
\langle 3\rangle &= \langle 3, a+\sqrt{m}\rangle \langle 3, a-\sqrt{m}\rangle = \mathcal{P}_5 \bar{\mathcal{P}}_5, \text{ where } a^2 \equiv m \pmod 3),
\end{aligned}$$

since $x^2 \equiv m \pmod 3$ is solvable. In the ring $\mathcal{O}_{M_3}$, we have

$$\begin{aligned}
\langle 2\rangle &= \mathcal{P}_6, \\
\langle 3\rangle &= \langle 3, \sqrt{m_1 n_1}\rangle^2 = \mathcal{P}_7^2, \text{ if } c \equiv 3, 9 \pmod{12}, \\
\langle 3\rangle &= \mathcal{P}_8, \text{ if } c \equiv 1, 7 \pmod{12}.
\end{aligned}$$

Using primitive solutions f_i, $i = 1, 2, 3$ of (7) we can rewrite system (3), (4) and (5) as

$$(cV)^2 - nU^2 = \pm s_1 P \tag{21}$$

$$((c+4)V)^2 - mZ^2 = \pm s_2 P \tag{22}$$

$$(cZ)^2 - m_1 n_1 U^2 = \pm s_3 P \tag{23}$$

with $s_1 = cf_1$, $s_2 = (c+4) f_2$, $s_3 = 4cf_3$ and $P = 2^A 3^B$. Let (U, V, Z) arbitrary but fixed solution of system (21), (22) and (23). Then, following [8, Section 6], we set

i	α_i	β_i	φ_{1i}	φ_{2i}	D_{1i}	D_{2i}
1	c	$\sqrt{n}$	$cV - \sqrt{n}U$	$cV + \sqrt{n}U$	0	3
2	$c+4$	$\sqrt{m}$	$(c+4)V - \sqrt{m}Z$	$(c+4)V + \sqrt{m}Z$	1	3
3	c	$\sqrt{m_1 n_1}$	$cZ - \sqrt{m_1 n_1}U$	$cZ + \sqrt{m_1 n_1}U$	0	2

and by [8, Lemma 3] we obtain the following:

- Ideal $\langle 2\rangle = \mathcal{P}_6$ is prime in the ring of integers $\mathcal{O}_{M_3}$ of the field M_3. Since $c \equiv c+4 \equiv 1 \pmod 2$, we have $ord_{\mathcal{P}_6}(2c) = 1$ and $ord_{\mathcal{P}_6}\left(2\sqrt{c(c+4)}\right) = 1$, and by [8, Lemma 3, (ii)], we obtain

  $A \leq 2\max\left\{ord_{\mathcal{P}_6}(2c), ord_{\mathcal{P}_6}\left(2\sqrt{c(c+4)}\right)\right\} + D_{23} = 2 \cdot \max\{1, 1\} + 2 = 4;$

- Let $c \equiv 1,7 \pmod{12}$. In this case ideal $\langle 3 \rangle = \mathcal{P}_3$ is prime in the ring of integers $\mathcal{O}_{M_1}$ of the field M_1. Since $c \equiv 1 \pmod 3$ and $c(c-2) \equiv 2 \pmod 3$ we obtain $ord_{\mathcal{P}_3}(2c) = 0$ and $ord_{\mathcal{P}_3}\left(2\sqrt{c(c-2)}\right) = 0$, and by [8, Lemma 3, (i)], we have

$$B \le 2\max\left\{ord_{\mathcal{P}_3}(2c), ord_{\mathcal{P}_3}\left(2\sqrt{c(c-2)}\right)\right\} + D_{11} = 0, \quad \text{i.e. } B = 0;$$

- Let $c \equiv 3,9 \pmod{12}$. In this case we have $\langle 3 \rangle = \left\langle 3, \sqrt{c(c-2)}\right\rangle^2 = \mathcal{P}_2^2$ in the ring of integers $\mathcal{O}_{M_1}$ of the field M_1. Since we have $3 \,\|\, c$ and $3 \nmid (c-2)$, then $ord_{\mathcal{P}_2}(2c) = 2$ and $ord_{\mathcal{P}_2}\left(2\sqrt{c(c-2)}\right) = 1$, and by [8, Lemma 3, (iii)], we obtain

$$B \le \max\left\{ord_{\mathcal{P}_2}(2c), ord_{\mathcal{P}_2}\left(2\sqrt{c(c-2)}\right)\right\} + D_{11} = \max\{2,1\} + 0 = 2.$$

Hence, we obtain:

- If $c \equiv 1,7 \pmod{12}$, than $\gcd(F_1, F_2, F_3) = 2^A$, where $0 \le A \le 4$;
- If $c \equiv 3,9 \pmod{12}$, than $\gcd(F_1, F_2, F_3) = 2^A 3^B$, where $0 \le A \le 4$ and $0 \le B \le 2$.

4.3 Upper bound for the exponents

Let us denote $\theta_1 = c+4$, $\theta_2 = c-2$ and $\theta_3 = c_1$, where $c_1 = c$ if $c \equiv 1,7 \pmod{12}$ and $c_1 = c/3$ if $c \equiv 3,9 \pmod{12}$. Then $ord_2(\theta_l) = ord_3(\theta_l) = 0$ for all $l = 1,2,3$. We assumed that in (20) exactly one d_i' is positive, exactly one e_j' is positive and $i \ne j$. Therefore, let $d_i' \ne 0$ and $e_j' \ne 0$, where $i \ne j$. Then, from (20), for distinct integers $i,j,k \in \{1,2,3\}$, we obtain

$$1 \le d_i' = ord_2\left(\pm\theta_j 3^{e_j'} \mp \theta_k 3^{e_k'}\right) = ord_2\left(3^{e_j'} - \left(\pm\frac{\theta_k}{\theta_j}\right)\right)$$
$$1 \le e_j' = ord_3\left(\pm\theta_i 2^{d_i'} \mp \theta_k 2^{d_k'}\right) = ord_3\left(2^{d_i'} - \left(\pm\frac{\theta_k}{\theta_i}\right)\right)$$

In all above cases we have expressions of the form

$$ord_p\left(\alpha_1^{b_1} - \alpha_2\right), \tag{24}$$

where b_1 is positive integer and $\alpha_1, \alpha_2 \in \mathbb{Q}$. We will apply estimates of Y. Bugeaud and M. Laurent [2, Corollaire 2] on (24).

Let us denote by $h(\alpha)$ the absolute logarithmic height of the algebraic number α, given by $h(0) = 0$ and by

$$h(\alpha) = \frac{1}{d}\log\left(|a|\prod_{l=1}^{d}\max\{1, |\alpha_l|\}\right)$$

if $a(x-\alpha_1)\dots(x-\alpha_d)$ is the minimal polynomial of $\alpha \neq 0$ over $\mathbb{Z}$. We have $h(2)=\log 2$, $h(3)=\log 3$, and

$$h\left(\pm\frac{\theta_k}{\theta_t}\right)=\frac{1}{1}\log\left(\theta_t\cdot\prod_{l=1}^{1}\max\left\{1,\left|\pm\frac{\theta_k}{\theta_t}\right|\right\}\right)=\log\left(\max\left\{\theta_t,\theta_k\right\}\right),$$

where $t=i$ or j. Therefore, if $\theta_k=\theta_1$ or $\theta_t=\theta_1$, then

$$h\left(\pm\frac{\theta_k}{\theta_t}\right)=\log\left(c+4\right).$$

If $\frac{\theta_k}{\theta_t}=\frac{c-2}{c_1}$ or $\frac{c_1}{c-2}$, then

$$h\left(\pm\frac{\theta_k}{\theta_t}\right)=\begin{cases}\log\left(c-2\right), & \text{if } c\equiv 3,9\,(\mathrm{mod}\,12),\\ \log c_1=\log c, & \text{if } c\equiv 1,7\,(\mathrm{mod}\,12).\end{cases}$$

Using the notations from [2], we have $K=\mathbb{Q}\left(\pm\frac{\theta_k}{\theta_t},2\right)=\mathbb{Q}\left(\pm\frac{\theta_k}{\theta_t},3\right)=\mathbb{Q}$, $f=1$ and $D=\frac{[K:\mathbb{Q}]}{f}=1$. Let $A_1>1$ and $A_2>1$ be real numbers such that

$$\max\left\{h\left(\alpha_i\right),\frac{\log p}{D}\right\}\leq\log A_i,\ \ i=1,2.$$

In our case we have

$$\max\left\{h\left(\alpha_1\right),\frac{\log p}{D}\right\}=\log 3,$$

$$\max\left\{h\left(\alpha_2\right),\frac{\log p}{D}\right\}=\max\left\{\log\max\left\{\theta_k,\theta_t\right\},\log p\right\}\leq\log\left(c+4\right),$$

so we can take $\log A_1=\log 3$ and $\log A_2=\log(c+4)$. Now, we have

$$b'=\frac{b_1}{D\log A_2}+\frac{b_2}{D\log A_1}=\frac{b_1}{\log\left(c+4\right)}+\frac{1}{\log 3}.$$

where $b_1=d_i'$ if $p=3$ and $b_1=e_j'$ if $p=2$. Hence, if $c\geq 3$, then [2, Corolllaire 2] implies

$$\begin{aligned}d_i' &\leq \frac{48\log 3}{(\log 2)^4}\left(\max\left\{\log\left(\frac{e_j'}{\log\left(c+4\right)}+\frac{1}{\log 3}\right)+\log\log 2+0.4,\ 10\right\}\right)^2\log\left(c+4\right)\\ &\leq \frac{48\log 3}{(\log 2)^4}\left(\max\left\{\log\left(\frac{e_j'}{\log 7}+\frac{1}{\log 3}\right)+\log\log 2+0.4,\ 10\right\}\right)^2\log\left(c+4\right)\end{aligned}\tag{25}$$

and

$$\begin{aligned}e_j' &\leq \frac{36}{(\log 3)^3}\left(\max\left\{\log\left(\frac{d_i'}{\log\left(c+4\right)}+\frac{1}{\log 3}\right)+\log\log 3+0.4,10\log 3\right\}\right)^2\log\left(c+4\right)\\ &\leq \frac{36}{(\log 3)^3}\left(\max\left\{\log\left(\frac{d_i'}{\log 7}+\frac{1}{\log 3}\right)+\log\log 3+0.4,10\log 3\right\}\right)^2\log\left(c+4\right).\end{aligned}\tag{26}$$

Let $3 \leq c \leq C_0$ and $T(c) = \max\{d_1', d_2', d_3', e_1', e_2', e_3'\} = \max\{d_i', e_j'\}$.
If $T(c) = d_i'$, then from (25) we have

$$e_j' \leq d_i' \leq \begin{cases} 22845 \cdot \log(C_0+4), & \text{if } e_j' \leq 41448 \\ 228.447 \left(\log\left(\frac{e_j'}{\log 7} + \frac{1}{\log 3}\right) + \log\log 2 + 0.4\right)^2 \log(C_0+4), & \text{if } e_j' \geq 41449. \end{cases} \tag{27}$$

If $T(c) = e_j'$, then from (26) we obtain

$$d_i' \leq e_j' \leq \begin{cases} 3276.87 \cdot \log(C_0+4), & \text{if } d_i' \leq 70107 \\ 27.16 \left(\log\left(\frac{d_i'}{\log 7} + \frac{1}{\log 3}\right) + \log\log 3 + 0.4\right)^2 \log(C_0+4) & \text{if } d_i' \geq 70108. \end{cases} \tag{28}$$

Therefore, if $3 \leq c \leq C_0$, then $T(c) \leq T_0(C_0)$, where $T_0(C_0)$ we can obtain from inequalities (27) and (28). Note, for calculating $T_0(C_0)$ it is enough to consider inequality

$$x \leq 228.447 \cdot \left(\log\left(\frac{x}{\log 7} + \frac{1}{\log 3}\right) + \log\log 2 + 0.4\right)^2 \cdot \log(C_0+4). \tag{29}$$

Indeed, if inequality (29) implies $x \leq K_0$, then $T(c) \leq K_0$, so we take $T_0(C_0) = K_0$. For example, if $c \leq C_0 = 100$, then from inequality (29) we obtain $T(c) \leq T_0(100) = 132125$. Similarly, we find $T_0(10^{10}) = 899597$ and $T_0(3) = 45234$.

Since estimates of Y. Bugeaud and M. Laurent [2, Corollaire 2] give a large upper bound for the exponents in (20), we can diminish that upper bound using [6, Lemma 4.1]. Note that (24) is easy to convert into expressions with $p-$adic logarithms. Namely, by [13, Lemma II.9], we have $d_i' = 1$ or

$$\begin{aligned} 2 \leq d_i' &= ord_2\left(3^{e_j'} - \left(\pm\frac{\theta_k}{\theta_j}\right)\right) = ord_2\left(\left(\pm\frac{\theta_j}{\theta_k}\right) \cdot 3^{e_j'} - 1\right) \\ &= ord_2\left(\log_2\left(\pm\frac{\theta_j}{\theta_k}\right) + e_j' \log_2 3\right) \end{aligned} \tag{30}$$

and

$$\begin{aligned} 1 \leq e_j' &= ord_2\left(2^{d_i'} - \left(\pm\frac{\theta_k}{\theta_i}\right)\right) = ord_2\left(\left(\pm\frac{\theta_i}{\theta_k}\right) \cdot 2^{d_i'} - 1\right) \\ &= ord_3\left(\log_3\left(\pm\frac{\theta_i}{\theta_k}\right) + d_i' \log_3 2\right) \end{aligned} \tag{31}$$

since $\left(\pm\frac{\theta_i}{\theta_k}\right) \cdot 3^{e_j'}$ is $2-$adic unit in Ω_2 and $\left(\pm\frac{\theta_j}{\theta_k}\right) \cdot 2^{d_i'}$ is $3-$adic unit in Ω_3 for all distinct integers $i, j, k \in \{1, 2, 3\}$. By repeating the $p-$adic reduction procedure given in [6, Lemma 4.1] for linear forms in $p-$adic logarithms from (30) and (31) as long as the reduced bounds are less than the original one, for each c, $3 \leq c \leq C_0$, we can obtain

$$d_i' := d_i'(c) \leq M_R^{(i)}(c) \quad \text{and} \quad e_j' := e_j'(c) \leq N_R^{(j)}(c),$$

where $M_R^{(i)}(c)$ and $N_R^{(j)}(c)$ are the best possible bounds for d'_i and e'_j, respectively. Denote

$$M_R(C_0) = \max_{i\in\{1,2,3\},\ c\leq C_0} M_R^{(i)}(c) \quad \text{and} \quad N_R(C_0) = \max_{j\in\{1,2,3\},\ c\leq C_0} N_R^{(j)}(c),$$

(where four exceptional cases for $c = 3$ given in Section 4.1 are also included). Then

$$d'_i \leq M_R(C_0) \leq T_0(C_0) \quad \text{and} \quad e'_i \leq N_R(C_0) \leq T_0(C_0),$$

for all $3 \leq c \leq C_0$ and all $i, j \in \{1, 2, 3\}$. The results from Section 4.1 now imply that the values of d_1, d_2, d_3, e_1, e_2, e_2 are also bounded which again implies, together with the results from Section 4.2, that values of the $F_i = 2^{d_i+A}3^{e_i+B}$, $i = 1, 2, 3$ are bounded too. Precisely, we obtain

$$ord_2(F_1F_2F_3) = \sum_{i=1}^{3} d_i + 3A \leq (M_R(C_0) + 4) + 3\cdot 4 = M_R(C_0) + 16 = M_0(C_0),$$

and

$$ord_3(F_1F_2F_3) = \sum_{j=1}^{3} e_j + 3B$$

$$\leq \begin{cases} N_R(C_0) = N_0(C_0), & \text{if } c \equiv 1, 7 \pmod{12}, \\ (N_R(C_0) + 2) + 3\cdot 2 = N_R(C_0) + 8 = N_0(C_0), & \text{if } c \equiv 3, 9 \pmod{12}. \end{cases}$$

Note that reduction procedure can be used only for particular values of the parameter c, so, unfortunately, reduced upper bounds $M_R(C_0)$ and $N_R(C_0)$ are not effectively computable constants if C_0 is too large. Therefore, if we put effectively computable constant $T_0(C_0)$ instead $M_R(C_0)$ and $N_R(C_0)$ in above formulas for $M_0(C_0)$ and $N_0(C_0)$, we have proved Theorem 2.

4.4 The reduction procedure

Let $3 \leq c \leq C_0$. Suppose

$$T(c) = \max\{d'_1, d'_2, d'_3, e'_1, e'_2, e'_3\} = \max\{d'_i, e'_j\} \leq T_0(C_0).$$

where $d'_i \neq 0$, $e'_j \neq 0$ and $i \neq j$. We consider linear forms

$$\Lambda_2 = \log_2\left(\pm\frac{\theta_j}{\theta_k}\right) + e'_j \log_2 3 \quad \text{and} \quad \Lambda_3 = \log_3\left(\pm\frac{\theta_i}{\theta_k}\right) + d'_i \log_3 2, \qquad (32)$$

where i, j, k are distinct integers from the set $\{1, 2, 3\}$ and $\theta_1 = c+4$, $\theta_2 = c-2$, $\theta_3 = c_1$. We can diminish the upper bound $T_0(C_0)$ applying [6, Lemma 4.1] on linear forms in (32). Using the notations from [6, Lemma 4.1] we have $n = 2$, $p = 2$ or 3,

$$X = \max\{|x_1|, |x_2|\} = \max\{1, e'_j\} = e'_j \leq T(c) \leq T_0(C_0) = X_0 \text{ if } p = 2,$$
$$X = \max\{|x_1|, |x_2|\} = \max\{1, d'_j\} = d'_i \leq T(c) \leq T_0(C_0) = X_0 \text{ if } p = 3,$$

$$\vartheta_1 = \log_2\left(\pm\frac{\theta_j}{\theta_k}\right),\ \vartheta_2 = \log_2 3 \ \text{ if } p = 2,$$
$$\vartheta_1 = \log_3\left(\pm\frac{\theta_i}{\theta_k}\right),\ \vartheta_2 = \log_3 2 \ \text{ if } p = 3.$$

Then we have

$$ord_2(\Lambda_2) = d_i' \geq e_j' = 0 + 1\cdot e_j' \ \text{ if } \ T(c) = d_i',$$
$$ord_3(\Lambda_3) = e_j' \geq d_i' = 0 + 1\cdot d_i' \ \text{ if } \ T(c) = e_j'.$$

Therefore, constants c_1 and c_2 from the [6, Lemma 4.1] are given by $(c_1, c_2) = (0, 1)$. Since

$$ord_2\left(\log_2\left(\pm\frac{\theta_j}{\theta_k}\right)\right) \geq ord_2(\log_2 3) = 2,$$
$$ord_3\left(\log_3\left(\pm\frac{\theta_i}{\theta_k}\right)\right) \geq ord_3(\log_3 2) = 1,$$

for all distinct integers i, j, k from the set $\{1, 2, 3\}$ (see below), then, following [6, Lemma 4.1], we define

$$\Lambda_2' = \frac{\Lambda_2}{\log_2 3} = -\left(-\frac{\log_2\left(\pm\frac{\theta_j}{\theta_k}\right)}{\log_2 3}\right) + e_j' = -(\vartheta_{j,k}) + e_j',\ \text{if } e_j' \geq 1 \text{ and } p = 2, \tag{33}$$

$$\Lambda_3' = \frac{\Lambda_3}{\log_3 2} = -\left(-\frac{\log_3\left(\pm\frac{\theta_i}{\theta_k}\right)}{\log_3 2}\right) + d_i' = -(\vartheta_{i,k}) + d_i',\ \text{if } d_i' \geq 2 \text{ and } p = 3. \tag{34}$$

For $0 < \mu_p \in \mathbb{Z}$ and $\vartheta_{t,k} \in \Omega_p$ let $\vartheta_{t,k}^{(\mu_p)}$ be a unique rational integer with $ord_2(\vartheta_{t,k} - \vartheta_{t,k}^{(\mu_p)}) \geq \mu_p$ and $0 \leq \vartheta_{t,k}^{(\mu_p)} \leq p^{\mu_p} - 1$, where $t = j$ if $p = 2$ and $t = i$ if $p = 3$. Denote by Γ_{μ_p} the lattice spanned by the columns of the matrix

$$\begin{bmatrix} 1 & 0 \\ \vartheta_{t,k}^{(\mu_p)} & p^{\mu_p} \end{bmatrix}.$$

Let

$$N_0^{(j)}(c) = \frac{\mu_2 - 1 + ord_2(\log_2 3) - 0}{1} = \mu_2 + 1$$

and

$$M_0^{(i)}(c) = \frac{\mu_3 - 1 + ord_3(\log_3 2) - 0}{1} = \mu_3.$$

Denote by $b_1^{(p)}$ the first vector of the $LLL-$reduced basis of Γ_{μ_p}. If

$$\left\| b_1^{(p)} \right\| > 2^{\frac{n-1}{2}}\sqrt{2}\cdot T_0(C_0) = 2\cdot T_0(C_0),$$

then, by [6, Lemma 4.1] we have:

- If $T(c) = d_i'$ there is no e_j' with $N_0^{(j)}(c) \leq e_j' \leq T_0(C_0)$.

- If $T(c) = e_j'$ there is no d_i' with $M_0^{(i)}(c) \leq d_i' \leq T_0(C_0)$.

Using these new bounds, the reduction can be repeated, as long as the new bound for e_j' or d_i' is less than the previous one. Finally, we obtain:

- If $T(c) = d_i'$, then $1 \leq e_j' \leq N_R^{(j)}(c)$, where $N_R^{(j)}(c)$ is the best possible bound for e_j' and all possible values for (d_i', e_j') we obtain from equation $2^{d_i'} = \pm \frac{\theta_j 3^{e_j'} \pm \theta_k}{\theta_i}$, where $1 \leq e_j' \leq N_R^{(j)}(c)$ and $d_i' \geq e_j'$.

- Similarly, if $T(c) = e_j'$, then $1 \leq d_i' \leq M_R^{(i)}(c)$, where $M_R^{(i)}(c)$ is the best possible bound for d_i' and all possible values for (d_i', e_j'), we obtain from equation $3^{e_j'} = \pm \frac{\theta_i 2^{d_i'} \pm \theta_k}{\theta_j}$, with $1 \leq d_i' \leq M_R^{(i)}(c)$ and $e_j' \geq d_i'$.

In order to find all possible elements with an index of the form $2^a 3^b$ for given c, $3 \leq c \leq C_0$, we have to perform above reduction procedure for each of six possible couples (d_i', e_j'), where $i \neq j$. Having determined all possible sextuples $(d_1', d_2', d_3', e_1', e_2', e_3')$, we have to find all possible sextuples $(d_1, d_2, d_3, e_1, e_2, e_3)$ (a connection between them is given in Section 4.1) which give us all possible primitive triples (f_1, f_2, f_3), where $f_i = 2^{d_i} 3^{e_i}$, $i = 1, 2, 3$. After that, all solutions of equation (7) which are of the form $(\pm f_1, \pm f_2, \pm f_3)$, we obtain using direct testing. Now all required triples $(\pm F_1, \pm F_2, \pm F_3)$ are of the form $\pm F_i = \pm f_i 2^A 3^B$, $i = 1, 2, 3$, where $0 \leq A \leq 4$ and $B = 0$ if $c \equiv 1, 7 \pmod{12}$ or $0 \leq B \leq 2$ if $c \equiv 3, 9 \pmod{12}$. For each explicit value of the triple $(\pm F_1, \pm F_2, \pm F_3)$ we have to solve a corresponding system (3), (4) and (5). Every solution (U, V, Z) of that system which is of the form (6), where $\gcd(x_2, x_3, x_4) = 1$, determines an integral element α of the form (2) with index $\mu(\alpha) = F_1 F_2 F_3$. For $3 \leq c \leq C_0$, we can give solutions in the form $(c;\ \mu(\alpha);\ \alpha) = (c;\ 2^a 3^b;\ x_2, x_3, x_4)$.

Computing $ord_p\left(\log_p\left(\pm\frac{\theta_t}{\theta_k}\right)\right)$ and $\vartheta_{t,k}^{(\mu_p)}$

From a definition of $p-$adic logarithm follows that, in our case, it is enough to find

$$\log_p \frac{\theta_t}{\theta_k} = \log_p \frac{c-2}{c+4}, \quad \log_p \frac{c-2}{c_1} \quad \text{and} \quad \log_p \frac{c_1}{c+4}$$

for $p = 2$ and $p = 3$ since

$$\log_p \frac{\theta_k}{\theta_t} = -\log_p \frac{\theta_t}{\theta_k} \quad \text{and} \quad \log_p\left(-\frac{\theta_t}{\theta_k}\right) = \log_p \frac{\theta_t}{\theta_k}.$$

Also, we have $ord_p\left(\log_p \frac{\theta_t}{\theta_k}\right) = ord_p\left(-\log_p \frac{\theta_t}{\theta_k}\right)$.

Using presentation of the $p-$adic logarithm as Taylor series, first we find

$$\log_2 3 = \frac{1}{2}\log_2\left(1-\left(-2^3\right)\right) = \sum_{n=0}^{\infty}(-1)^n\,\frac{2^{3n+2}}{n+1},$$
$$\log_3 2 = \frac{1}{2}\log_3\left(1-(-3)\right) = \sum_{n=0}^{\infty}(-1)^n\,\frac{3^{n+1}}{2\,(n+1)},$$

which implies $ord_2\left(\log_2 3\right)=2$ and $ord_3\left(\log_3 2\right)=1$, respectively.

Similarly, we find all $\log_p \frac{\theta_t}{\theta_k}$ for $p=2$ and $p=3$ from which it follows that

$$ord_2\left(\log_2\left(\pm\frac{\theta_t}{\theta_k}\right)\right) \geq ord_2\left(\log_2 3\right)=2,$$
$$ord_3\left(\log_3\left(\pm\frac{\theta_t}{\theta_k}\right)\right) \geq ord_3\left(\log_3 2\right)=1.$$

for all $\frac{\theta_t}{\theta_k}=\frac{c-2}{c+4},\frac{c-2}{c_1},\frac{c_1}{c+4}$. Therefore, since we follow [6, Lemma 4.1], the $p-$adic integers $\vartheta_{j,k}$ and $\vartheta_{i,k}$ in (33) and (34) have to be defined as: $\vartheta_{j,k}=-\frac{\log_2\left(\pm\frac{\theta_j}{\theta_k}\right)}{\log_2 3}$ if $p=2$ and $\vartheta_{i,k}=-\frac{\log_3\left(\pm\frac{\theta_i}{\theta_k}\right)}{\log_3 2}$ if $p=3$.

Additionally, using the presentation of the $p-$adic logarithm as Taylor series, each $\alpha:=\vartheta_{t,k}$ we can rewrite in the form

$$\alpha=\frac{\pm\sum\limits_{n=0}^{\infty}a_n}{\sum\limits_{n=0}^{\infty}b_n}=\frac{\alpha_1}{\alpha_2},$$

where $ord_p\left(\alpha_1\right)\geq 0$ and $ord_p\left(\alpha_2\right)=0$. Then, for every $0<\mu\in\mathbb{Z}$, we can find sufficiently large integers n_1 and n_2 such that $\alpha_1'=\pm\sum\limits_{n=0}^{n_1}a_n$ and $\alpha_2'=\sum\limits_{n=0}^{n_2}b_n$ satisfy $\alpha^{(\mu)}\equiv\frac{\alpha_1'}{\alpha_2'}\left(\bmod p^\mu\right)$, where $\alpha^{(\mu)}$ is a unique rational integer with $ord_p(\alpha-\alpha^{(\mu)})\geq\mu$ and $0\leq\alpha^{(\mu)}\leq p^\mu-1$. We obtain the following

$$\alpha^{(\mu)}\equiv\frac{\pm\log_2\frac{\theta_t}{\theta_n}}{\log_2 3}\left(\bmod 2^\mu\right)\equiv\frac{\pm\sum\limits_{n=0}^{n_1}a_n}{\sum\limits_{n=0}^{n_2}(-1)^n\,\frac{2^{3n}}{n+1}}\left(\bmod 2^\mu\right),$$

where $n_2=\left\lfloor\frac{\mu-1}{2}\right\rfloor$ and

$$\alpha^{(\mu)}\equiv\frac{\pm\log_3\frac{\theta_t}{\theta_n}}{\log_3 2}\left(\bmod 3^\mu\right)\equiv\frac{\pm\sum\limits_{n=0}^{n_1}a_n}{\sum\limits_{n=0}^{n_2}(-1)^n\,\frac{3^n}{2(n+1)}}\left(\bmod 3^\mu\right),$$

where $n_2 = 2\mu - 3$ if $\mu \geq 3$, and $n_2 = 2$ if $\mu = 1, 2$. The values of $\pm\sum\limits_{n=0}^{n_1} a_n$ are given in the following two tables:

1) Case $c \equiv 1, 7 \,(\mathrm{mod}\, 12)$. Note, in this case we have $c_1 = c$.

$\alpha^{(\mu)} \equiv \alpha \,(\mathrm{mod}\, p^\mu)$	$0 \leq l \leq \mu - 1$	$n_1 = n_1^{(l)}$	$\alpha_1' = \pm\sum\limits_{n=0}^{n_1} a_n$
$\frac{\pm \log_2\left(\frac{c-2}{c_1}\right)}{\log_2 3} \,(\mathrm{mod}\, 2^\mu)$	$ord_2\left(\frac{c-1}{2}\right)$	$\left\lfloor \frac{\mu-l-1}{l+2} \right\rfloor$	$\pm\sum\limits_{n=0}^{n_1} \frac{-2^{3k}\left(\frac{c-1}{2}\right)^{n+1}}{(n+1)c^{2k+2}}$
$\frac{\pm \log_2\left(\frac{c-2}{c+4}\right)}{\log_2 3} \,(\mathrm{mod}\, 2^\mu)$	$ord_2\left(\frac{c+1}{2}\right)$	$\left\lfloor \frac{\mu-l-1}{l+2} \right\rfloor$	$\pm\sum\limits_{n=0}^{n_1} \frac{2^{3k}3^{n+1}\left(\frac{c+1}{2}\right)^{n+1}}{(n+1)(c+4)^{2k+2}}$
$\frac{\pm \log_2\left(\frac{c_1}{c+4}\right)}{\log_2 3} \,(\mathrm{mod}\, 2^\mu)$	-	$\mu - 1$	$\pm\sum\limits_{n=0}^{n_1} - \frac{2^{2k}}{(n+1)(c+4)^{n+1}}$
$\frac{\pm \log_3\left(\frac{c-2}{c_1}\right)}{\log_3 2} \,(\mathrm{mod}\, 3^\mu)$	$ord_3\left(\frac{c-1}{3}\right)$	$\left\lfloor \frac{\mu-l-1}{l} \right\rfloor$, if $l > 0$ $2\mu - 3$, if $l = 0, \mu \geq 3$ 2, if $l = 0$, $\mu = 1, 2$	$\pm\sum\limits_{n=0}^{n_1} - \frac{2^{2k+1}3^n\left(\frac{c-1}{3}\right)^{n+1}}{(n+1)c^{2k+2}}$
$\frac{\pm \log_3\left(\frac{c-2}{c+4}\right)}{\log_3 2} \,(\mathrm{mod}\, 3^\mu)$	-	$2\mu - 3$, if $\mu \geq 3$ 2, if $\mu = 1, 2$	$\pm\sum\limits_{n=0}^{n_1} - \frac{2^{n+1}3^n}{(n+1)(c+4)^{n+1}}$
$\frac{\pm \log_3\left(\frac{c_1}{c+4}\right)}{\log_3 2} \,(\mathrm{mod}\, 3^\mu)$	$ord_3\left(\frac{c+2}{3}\right)$	$\left\lfloor \frac{\mu-l-1}{l} \right\rfloor$, if $l > 0$ $2\mu - 3$, if $l = 0, \mu \geq 3$ 2, if $l = 0$, $\mu = 1, 2$	$\pm\sum\limits_{n=0}^{n_1} - \frac{2^{3k+3}3^n\left(\frac{c+2}{3}\right)^{n+1}}{(n+1)(c+4)^{2k+2}}$

2) Case $c \equiv 3, 9 \,(\mathrm{mod}\, 12)$. Note, in this case we have $c_1 = \frac{c}{3}$.

$\alpha^{(\mu)} \equiv \alpha \,(\mathrm{mod}\, p^\mu)$	$0 \leq l \leq \mu - 1$	$n_1 = n_1^{(l)}$	$\alpha_1' = \pm\sum\limits_{n=0}^{n_1} a_n$
$\frac{\pm \log_2\left(\frac{c-2}{c_1}\right)}{\log_2 3} \,(\mathrm{mod}\, 2^\mu)$	$ord_2\left(\frac{c-3}{2}\right)$	$\left\lfloor \frac{\mu-l-1}{l+1} \right\rfloor$	$\pm\sum\limits_{n=0}^{n_1} (-1)^n \frac{2^{2k}\left(\frac{c-3}{2}\right)^{n+1}}{(n+1)c^{n+1}}$
$\frac{\pm \log_2\left(\frac{c-2}{c+4}\right)}{\log_2 3} \,(\mathrm{mod}\, 2^\mu)$	$ord_2\left(\frac{c+1}{2}\right)$	$\left\lfloor \frac{\mu-l-1}{l+2} \right\rfloor$	$\pm\sum\limits_{n=0}^{n_1} - \frac{2^{3k}3^{n+1}\left(\frac{c+1}{2}\right)^{n+1}}{(n+1)(c+4)^{2k+2}}$
$\frac{\pm \log_2\left(\frac{c_1}{c+4}\right)}{\log_2 3} \,(\mathrm{mod}\, 2^\mu)$	$ord_2\left(\frac{c+3}{2}\right)$	$\left\lfloor \frac{\mu-l-3}{l+4} \right\rfloor$	$\pm\sum\limits_{n=0}^{\infty} \frac{2^{4k+2}(c+6)^{n+1}\left(\frac{c+3}{2}\right)^{n+1}}{3^{2k+2}(n+1)(c+4)^{2k+2}}$
$\frac{\pm \log_3\left(\frac{c-2}{c_1}\right)}{\log_3 2} \,(\mathrm{mod}\, 3^\mu)$	$ord_3\left(\frac{c-3}{3}\right)$	$\left\lfloor \frac{\mu-l-1}{l} \right\rfloor$, if $l > 0$ $2\mu - 3$, if $l = 0$, $\mu \geq 3$ 2, if $l = 0$, $\mu = 1, 2$	$\pm\sum\limits_{n=0}^{n_1} (-1)^n \frac{2^{n+1}3^n\left(\frac{c-3}{3}\right)^{n+1}}{(n+1)c^{n+1}}$
$\frac{\pm \log_3\left(\frac{c-2}{c+4}\right)}{\log_3 2} \,(\mathrm{mod}\, 3^\mu)$	-	$2\mu - 3$, if $\mu \geq 3$ 2, if $\mu = 1, 2$	$\pm\sum\limits_{n=0}^{n_1} - \frac{2^{n+1}3^n}{(n+1)(c+4)^{n+1}}$
$\frac{\pm \log_3\left(\frac{c_1}{c+4}\right)}{\log_3 2} \,(\mathrm{mod}\, 3^\mu)$	$ord_3\left(\frac{(c+3)(c+6)}{9}\right)$	$\left\lfloor \frac{\mu-l}{l-1} \right\rfloor$, if $l \geq 2$ $2\mu - 3$, if $l = 1$, $\mu \geq 3$ 2, if $l = 1$, $\mu = 1, 2$	$\pm\sum\limits_{n=0}^{n_1} (-1)^n \frac{2^{3k+2}\left(\frac{(c+3)(c+6)}{9}\right)^{n+1}}{3(n+1)(c+4)^{2k+2}}$

If $l > \mu - 1$, then $\alpha^{(\mu)} \equiv \alpha \,(\mathrm{mod}\, p^\mu) \equiv 0$.

Taking the appropriate values of α_1' given in the tables above, we can calculate the values of $\vartheta_{t,k}^{(\mu_p)}$ at each step of the reduction procedure.

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