Computing relative power integral bases in a family of quartic extensions of imaginary quadratic fields

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Abstract

Let $M = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field with the ring of integers $\mathbb{Z}_M$ and let ξ be a root of polynomial $f(x) = x^4 - 2cx^3 + 2x^2 + 2cx + 1$, where $c \in \mathbb{Z}_M$, $c \notin \{0, \pm 2\}$. We consider an infinite family of octic fields $K_c = M(\xi)$ with the ring of integers $\mathbb{Z}_{K_c}$. Our goal is to determine all generators of relative power integral basis of $\mathcal{O} = \mathbb{Z}_M[\xi]$ over $\mathbb{Z}_M$. We show that our problem reduces to solving the system of relative Pellian equations

$$cV^2 - (c+2)U^2 = -2\mu, \quad cZ^2 - (c-2)U^2 = 2\mu,$$

where μ is an unit in $\mathbb{Z}_M$. We solve the system completely and find that all non-equivalent generators of power integral basis of $\mathcal{O}$ over $\mathbb{Z}_M$ are given by $\alpha = \xi$, $2\xi - 2c\xi^2 + \xi^3$ for $|c| \geq 159108$ and $|c| \leq 1000$, $c \notin S_c$ (where S_c is a set of exceptional cases, $|S_c| = 28$). Also, we find that, in all above cases, $\mathcal{O}$ admits no apsolute power integral basis if $-D \equiv 2, 3 \pmod 4$.

1 Introduction

Let K be an algebraic number field of degree n and $\mathbb{Z}_K$ its ring of integers. It is a classical problem in algebraic number theory to decide if K is *monogenic field*, or, equivalently, if K is a field for which there exist an element $\alpha \in \mathbb{Z}_K$ such that ring of integers $\mathbb{Z}_K$ is of the form $\mathbb{Z}_K = \mathbb{Z}[\alpha]$. The powers of such element α constitute a *power integral basis*, ie. an integral basis of the form $\{1, \alpha, \alpha^2, ..., \alpha^{n-1}\}$. In general, if $\{1, \omega_2, ..., \omega_n\}$ is an integral basis of K and the primitive integer $\alpha \in \mathbb{Z}_K$, $K = \mathbb{Q}(\alpha)$, is represented in that integral basis as $\alpha = x_1 + x_2\omega_2 + ... + x_n\omega_n$, then

$$I(\alpha) = \left[\mathbb{Z}_K^+ : \mathbb{Z}[\alpha]^+\right] = |I(x_2, ..., x_n)|,$$

where $\mathbb{Z}_K^+$ and $\mathbb{Z}[\alpha]^+$ respectively denote the additive groups of the ring $\mathbb{Z}_K$ and the polynomial ring $\mathbb{Z}[\alpha]$. The polynomial $I(X_2, ..., X_n)$ is a homogenous polynomial in $n-1$ variables of degree $\frac{n(n-1)}{2}$ with rational integer coefficients which is called the *index form*

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corresponding to the integral basis $\{1, \omega_2, ..., \omega_n\}$. The positive rational integer $I(\alpha)$ is called an *index* of the element α and it does not depend on x_1. Therefore, the primitive integer α generates a power integral basis if and only if $I(\alpha) = 1$. Consequently, the number field K is monogenic if and only if the *index form equation*

$$I(x_2, ..., x_n) = \pm 1 \tag{1}$$

is solvable in rational integers. The problem of determining all generators of the power integral basis reduces to the resolution of diophantine eq. (1).

The index form equations are mostly very complicated diophantine equations for number fields of large degree n. In some particular fields, by studying the structure of index form, it has been found a correspondence between the index form equation and simpler types of equations (for a survey see [6]). For example, in [8, 9], I. Gaál, A. Pethő and M. Pohst showed that a resolution of index form equations in any quartic field can be reduced to the resolution of cubic and several corresponding Thue equations. In [7], I. Gaál, and M. Pohst extended some basic ideas and developed a method of determining generators of a power integral basis to relative quartic extension fields K over base fields M. The method is much more complicated than in the absolute case. For example, instead of Thue equations we obtain relative Thue equations over a subfield M. The generalization of known methods to relative extensions leads to various nontrivial problems. Those problems occur primarily because a relative integral basis does not have to exist and also, the ring of integers of base field M is not necessarily an unique factorization domain.

Algorithms for solving index form equations have been applied in several infinite parametric families of certain fields. In particular, I. Gaál and T. Szabó in [10] considered three infinite parametric families of octic fields that are quartic extensions of imaginary quadratic fields. By applying the method described in [7] and by using results on infinite parametric families on relative Thue equations given in [15] and [12], they found all non-equivalent generators of relative power integral basis for infinite values of parameter.

In this paper, we consider the following problem. Let M be an imaginary quadratic field with the ring of integers $\mathbb{Z}_M$ and let ξ be a root of polynomial

$$f(x) = x^4 - 2cx^3 + 2x^2 + 2cx + 1,$$

where $c \in \mathbb{Z}_M$, $c \notin \{0, \pm 2\}$. We consider infinite family of octic fields $K_c = M(\xi)$ with ring of integers $\mathbb{Z}_{K_c}$. Since integral basis of K_c is not known in a parametric form, our goal is to determine all generators of relative power integral basis of $\mathcal{O} = \mathbb{Z}_M[\xi]$ over $\mathbb{Z}_M$ (instead of $\mathbb{Z}_{K_c}$ over $\mathbb{Z}_M$).

The paper is organized as follows. In Sections 2 and 3, we briefly describe the method of I. Gaál, and M. Pohst given in [7] and apply that method to the problem described above. In Section 4 we show that our problem reduces to solving the system of relative Pellian equations over M and apply some results given in [12]. In Section 5, by combining *congruence method* with an extension of Bennett's theorem given in [12], we solve the system completely and find all non-equivalent generators of power integral basis of $\mathcal{O}$ over $\mathbb{Z}_M$ if absolute value of parameter c is large enough ($|c| \geq 159108$). In Section 6 we assume that $|c| < 159108$ and apply a theorem of Baker and Wüstholz and a version of the reduction procedure due to Baker and Davenport. Without proving that the corresponding linear form $\Lambda \neq 0$, we cannot apply Baker's theory. The proof is rather complicated and involves several cases. We were not able to perform reduction procedure for all values of $|c| < 159108$ because we estimated that it would last more than 10^{10} sec. (in Mathematica on an simple PC). So, we have performed reduction procedure for $|c| \leq 1000$. Section 7 is devoted to the exceptional cases $c \in S_c$. For $c \in S_c$ at least one of the equation of our system of Pellian equations has additional classes of solutions or there exists only finitely many solutions of those equations. In the last section we examine whether the order $\mathcal{O} = \mathbb{Z}_M[\xi]$ admits an absolute power integral basis.

Our main result is the following theorem.

Theorem 1 *Assume that D is a square free positive integer, $M = \mathbb{Q}\left(\sqrt{-D}\right)$ is an imaginary quadratic with ring of integers $\mathbb{Z}_M$, ξ is a root of the polynomial*

$$f(t) = t^4 - 2ct^3 + 2t^2 + 2ct + 1,$$

where $c \in \mathbb{Z}_M$, $c \notin \{0, \pm 2\}$ and $K_c = M(\xi)$ is an octic field with ring of integers $\mathbb{Z}_{K_c}$. Then all non-equivalent generators of power integral basis of $\mathcal{O} = \mathbb{Z}_M[\xi]$ over $\mathbb{Z}_M$ are given by

$$\alpha = \xi, \quad 2\xi - 2c\xi^2 + \xi^3 \tag{2}$$

in each of the following cases:

i) *for all D and $|c| \geq 159108$;*

ii) *for all D, $c \notin S_c$ and $|c| \leq 1000$ or $\mathrm{Re}(c) = 0$;*

iii) *$c = \pm 1$ and $D = 1, 3$,*

where

$$S_c = \{\pm 1, \pm\sqrt{-1}, \pm 1 \pm \sqrt{-1}, \pm 2 \pm \sqrt{-1}, \pm 1 \pm \sqrt{-2}, \pm 1 \pm \sqrt{-3}, \frac{\pm 1 \pm \sqrt{-3}}{2}, \frac{\pm 3 \pm \sqrt{-3}}{2}\}, \tag{3}$$

with mixed signs.

Proof of Theorem 1. Immediately from propositions 14, 23, 24 and Corollary 21. ■

The current work supports the following conjecture.

Conjecture 2 *All non-equivalent generators of power integral basis of $\mathcal{O} = \mathbb{Z}_M[\xi]$ over $\mathbb{Z}_M$ are given by (2) for all D and $c \notin S_c$.*

Also, we prove the following theorem.

Theorem 3 *If $-D \equiv 2, 3 \pmod 4$ and all non-equivalent generators of power integral basis of $\mathcal{O} = \mathbb{Z}_M[\xi]$ over $\mathbb{Z}_M$ are given by (2), then $\mathcal{O}$ admits no absolute power integral basis. In particular, in the cases given in Theorem 1, $\mathcal{O}$ admits no absolute power integral basis.*

2 Preliminaries

Since we are going to apply the method of I. Gaál and M. Pohst given in [7], we begin with a brief description of it.

Let M be a field of degree m and K its quartic extension generated by an algebraic integer ξ over M, ie. $K = M(\xi)$. $\mathbb{Z}_K$ and $\mathbb{Z}_M$ denotes the ring of integers of K and M, respectively. Assume that a relative minimal polynomial of ξ is given by

$$f(t) = t^4 + a_1t^3 + a_2t^2 + a_3t + a_4 \in \mathbb{Z}_M[t].$$

Also, assume that d is the smallest natural number with the property $d\mathbb{Z}_K \subseteq \mathbb{Z}_M[\xi]$ and $i_0 = \left[\mathbb{Z}_K^+ : \mathbb{Z}_M[\xi]^+\right]$. Then each $\alpha \in \mathbb{Z}_K$ can be represented in the form

$$\alpha = \frac{1}{d}\left(a + x\xi + y\xi^2 + z\xi^3\right), \quad a, x, y, z \in \mathbb{Z}_M. \tag{4}$$

The (absolute) index of α can be factorized in the form

$$I(\alpha) = \left[\mathbb{Z}_K^+ : \mathbb{Z}_M[\alpha]^+\right]\left[\mathbb{Z}_M[\alpha]^+ : \mathbb{Z}[\alpha]^+\right]. \tag{5}$$

If the relative index $I_{K/M}(\alpha) = \left[\mathbb{Z}_K^+ : \mathbb{Z}_M[\alpha]^+\right]$ is equal to 1, then α can only generate a power integral basis in K (equivalently, $I(\alpha) = 1$). Let

$$F(u,v) = u^3 - a_2u^2v + (a_1a_3 - 4a_4)uv^2 + \left(4a_2a_4 - a_3^2 - a_1^2a_4\right)v^3 \tag{6}$$

be a binary cubic form over $\mathbb{Z}_M$, and

$$\begin{aligned} Q_1(x,y,z) &= x^2 - xya_1 + y^2a_2 + xz\left(a_1^2 - 2a_2\right) \\ &\quad + yz(a_3 - a_1a_2) + z^2\left(-a_1a_3 + a_2^2 + a_4\right), \end{aligned} \tag{7}$$

$$Q_2(x,y,z) = y^2 - xz - yza_1 + a_2z^2 \tag{8}$$

be ternary quadratic forms over $\mathbb{Z}_M$. In [7] the following assertion was proved. If $\alpha \in \mathbb{Z}_K$ given by (4) generates a relative power integral basis of $\mathbb{Z}_K$ over $\mathbb{Z}_M$, then there is a solution $(u,v) \in \mathbb{Z}_M^2$ of

$$N_{M/\mathbb{Q}}(F(u,v)) = \pm\frac{d^{6m}}{i_0}, \tag{9}$$

where

$$u = Q_1(x,y,z), \quad v = Q_2(x,y,z). \tag{10}$$

Note that the equation (9) implies

$$F(u,v) = \delta\varepsilon, \tag{11}$$

where δ is an integer in M of the norm $\pm d^{6m}/i_0$ and ε is an unit in M. Hence, the full set of nonassociated elements of this norm have to be considered.

In order to find all non-equivalent generators of power integral basis of $\mathbb{Z}_K$, the first step consists of solving the equation (11), ie. determining all (nonassociated) pairs $(u,v) \in \mathbb{Z}_M^2$ such that all solutions of (11) are of the form $(\eta u, \eta v)$, where $\eta \in M$ is an unit. In the next step, we have to find all $(x,y,z) \in \mathbb{Z}_M^3$ corresponding to a fixed solution (u,v) by solving the system (10). So, for a given solution (u,v) of (11), we solve the following equation

$$Q_0(x,y,z) = uQ_2(x,y,z) - vQ_1(x,y,z) = 0. \tag{12}$$

Using the arguments of Siegel [14, p.264] (see also [13]), it is possible to decide if (12) has nontrivial solutions and if so, all solutions of (12) can be given in a parametric form (with two parameters p and q). By substituting these parametric representations of u and v into the original system (10), it can be shown that at least one of the equations in (10) is a quartic Thue equation over $\mathbb{Z}_M$. By solving that Thue equation, we are able to determine all parameters $(p,q) \in \mathbb{Z}_M^2$ up to unit factors in M. Hence, we can calculate all $(x,y,z) \in \mathbb{Z}_M^3$ up to an unit factor of M, as well. Then all generators of power integral basis of $\mathbb{Z}_K$ over $\mathbb{Z}_M$ are of the form

$$\alpha = \frac{1}{d}\left(a + \eta\left(x\xi + y\xi^2 + z\xi^3\right)\right),$$

where $a \in \mathbb{Z}_M$ and the unit $\eta \in M$ are arbitrary. Consequently, all non-equivalent generators of power integral basis of $\mathbb{Z}_K$ over $\mathbb{Z}_M$ are given by $\alpha = \frac{1}{d}\left(x\xi + y\xi^2 + z\xi^3\right)$. For more details see [7].

Our purpose is to describe the relative power integral bases of either $\mathcal{O} = \mathbb{Z}_K$ over $\mathbb{Z}_M$ (if the integer basis of K is known) or of $\mathcal{O} = \mathbb{Z}_M[\xi]$ over $\mathbb{Z}_M$ (otherwise). Note that in the case $\mathcal{O} = \mathbb{Z}_M[\xi]$, ξ itself is a generator of a relative power integral basis but we wonder if there exist any other generators of power integral bases. Also, we have $i_0 = d = 1$. Consequently, equation (9) is of the form $F(u,v) = \varepsilon$, where ε is unit in M and non-equivalent generators of power integral basis of $\mathcal{O}$ over $\mathbb{Z}_M$ are of the form $\alpha = x\xi + y\xi^2 + z\xi^3$, where $(x,y,z) \in \mathbb{Z}_M^3$.

3 Resolution of relative cubic equation

Let D be a square free positive integer and let $M = \mathbb{Q}\left(\sqrt{-D}\right)$ be an imaginary quadratic with ring of integers $\mathbb{Z}_M$. Let ξ be a root of polynomial

$$f(t) = t^4 - 2ct^3 + 2t^2 + 2ct + 1, \tag{13}$$

where $c \in \mathbb{Z}_M$, $c \notin \{0, \pm 2\}$. We consider an infinite family of octic fields $K_c = M(\xi)$ with ring of integers $\mathbb{Z}_{K_c}$. It is easy to see that if $c = 0, \pm 2$, then $f(t)$ is a reducible polynomial and so K_c is not an octic field. Therefore, from now on we assume that $c \in \mathbb{Z}_M \setminus \{0, \pm 2\}$. Since the integral basis of K_c is not known in a parametric form, our goal is to determine all generators α of relative power integral basis of $\mathcal{O} = \mathbb{Z}_M[\xi]$ over $\mathbb{Z}_M$ (instead of $\mathbb{Z}_{K_c}$ over $\mathbb{Z}_M$). In this case the equation (11) is of the form

$$F(u,v) = (u + 2v)(u - 2(c+1)v)(u + 2(c-1)v) = \varepsilon, \tag{14}$$

where ε is an unit in M, ie. $\varepsilon \in \{\pm 1, \pm i, \pm\omega, \pm\omega^2\} \cap \mathbb{Z}_M$ and (7), (8) can be rewritten as

$$\begin{aligned} Q_1(x,y,z) &= x^2 + 2cxy + 2y^2 + 4\left(c^2 - 1\right)xz + 6cyz + z^2\left(4c^2 + 5\right) \\ Q_2(x,y,z) &= y^2 - xz + 2cyz + 2z^2. \end{aligned}$$

According to (14) we conclude that $u - 2v$, $u - 2(c+1)v$, $u + 2(c-1)v$ are units in $\mathbb{Z}_M$ and that implies $v = 0$. Therefore, all solutions of (14) are given by $(u,v) = (\eta, 0)$, where η is an unit in $\mathbb{Z}_M$.

4 Simultaneous Pellian equations

In this part we show that solving the equation (12) for $(u, v) = (\eta, 0)$ can be reduced to solving a system of simultaneous Pellian equations. Since $v = 0$, the equation (12) implies

$$Q_2(x, y, z) = y^2 - xz + 2cyz + 2z^2 = 0, \tag{15}$$

and $(x, y, z) = (2, 0, 1)$ is one nontrivial solution of (15). Therefore, all solutions can be parameterized by

$$x = 2r + p,\ y = q,\ z = r, \tag{16}$$

where $p, q, r \in M$ and $r \neq 0$. By substituting (16) into (15), we obtain

$$q^2 = r(p - 2cq). \tag{17}$$

Further, if we multiply (16) by $k = p - 2cq$, we get

$$kx = 2q^2 + p^2 - 2cqp,\ ky = qp - 2cq^2,\ kz = q^2. \tag{18}$$

We can assume that $k, p, q \in \mathbb{Z}_M$ and since the corresponding determinant equals 1, the parameter k must be an unit in $\mathbb{Z}_M$. Now, by substituting kx, ky, kz given by (18) into the equation $Q_1(x, y, z) = \eta$ (η is an unit in $\mathbb{Z}_M$) we obtain

$$p^4 - 2cp^3q + 2p^2q^2 + 2cpq^3 + q^4 = \mu, \tag{19}$$

where $\mu = k^2\eta$ is an unit in $\mathbb{Z}_M$. This is a relative Thue equation over $\mathbb{Z}_M$ and it can be transformed into a system of Pellian equations

$$cV^2 - (c + 2)U^2 = -2\mu, \tag{20}$$

$$(c - 2)U^2 - cZ^2 = -2\mu, \tag{21}$$

by putting

$$U = p^2 + q^2,\ V = p^2 + 2pq - q^2,\ Z = -p^2 + 2pq + q^2. \tag{22}$$

Both of equations (20) and (21) are of the same form as the equation already studied in [12], ie. of the form

$$(k - 1)x^2 - (k + 1)y^2 = -2\mu. \tag{23}$$

Proposition 4 ([12, Proposition 5.2]) *Let $k \in \mathbb{Z}_M$ and let $\mu \in \mathbb{Z}_M$ be an unit. Suppose $|k| \geq 2$ or k is not an element of the set*

$$S = \{0, \pm 1, \pm\sqrt{-1}, \pm 1 \pm \sqrt{-1}, \pm\sqrt{-2}, \pm\sqrt{-3}, \pm\omega, \pm\omega^2\},$$

with mixed signs, where $\omega = \frac{-1+\sqrt{-3}}{2}$. If the equation (23) is solvable, then

$$\mu \in \{1, -1, \omega, \omega^2\}.$$

All solutions are of the form $(x, y) = (\pm x_m, \pm y_m)$, with mixed signs, where the sequences (x_m) and (y_m) are given with the recurrence relations

$$x_0 = \varepsilon,\ x_1 = \varepsilon(2k + 1),\ x_{m+2} = 2kx_{m+1} - x_m,\ m \geq 0, \tag{24}$$

$$y_0 = \varepsilon,\ y_1 = \varepsilon(2k - 1),\ y_{m+2} = 2ky_{m+1} - y_m,\ m \geq 0, \tag{25}$$

where $\varepsilon = 1, \sqrt{-1}, \omega^2, \omega$ corresponds to $\mu = 1, -1, \omega, \omega^2$, respectively.

For $k=c+1$ Proposition 4 implies that if

$$c \notin \{-1, -1 \pm \sqrt{-1}, \pm\sqrt{-1}, -2\pm\sqrt{-1}, -1\pm\sqrt{-2}, -1\pm\sqrt{-3}, \frac{-1\pm\sqrt{-3}}{2}, \frac{-3\pm\sqrt{-3}}{2}\},$$

then all solutions (V,U) of (20) are of the form $(\pm v_m, \pm u_m)$, where

$$\begin{aligned} &v_0 = \varepsilon,\ v_1 = \varepsilon(2c+3),\ v_{m+2} = (2c+2)v_{m+1} - v_m,\ m \geq 0, \\ &u_0 = \varepsilon,\ u_1 = \varepsilon(2c+1),\ u_{m+2} = (2c+2)u_{m+1} - u_m,\ m \geq 0. \end{aligned} \tag{26}$$

Similarly, if $k=c-1$ and

$$c \notin \{1, 1\pm\sqrt{-1}, \pm\sqrt{-1}, 2\pm\sqrt{-1}, 1\pm\sqrt{-2}, 1\pm\sqrt{-3}, \frac{1\pm\sqrt{-3}}{2}, \frac{3\pm\sqrt{-3}}{2}\},$$

then all solutions (U,Z) of (21) are of the form $(\pm u'_n, \pm z_n)$, where

$$\begin{aligned} &u'_0 = \varepsilon,\ u'_1 = \varepsilon(2c-1),\ u'_{n+2} = (2c-2)u'_{n+1} - u'_n,\ n \geq 0, \\ &z_0 = \varepsilon,\ z_1 = \varepsilon(2c-3),\ z_{n+2} = (2c-2)z_{n+1} - u_n,\ n \geq 0. \end{aligned} \tag{27}$$

Finally, if $c \notin S_c$, where S_c is given in (3) and the system of equations (20) and (21) is solvable, then Proposition 4 implies

$$\mu \in \{1, -1, \omega, \omega^2\}.$$

Furthermore, if (U,V,Z) is a solution of that system, then

$$U = \pm u_m = \pm u'_n$$

for some $n,m \in \mathbb{N}_0$, with mixed signs, where u_m, u'_n are given by (26), (27) and $\varepsilon = 1, \sqrt{-1}, \omega^2, \omega$ corresponds to $\mu = 1, -1, \omega, \omega^2$. Evidently, $U = \pm u_0 = \pm u'_0 = \pm\varepsilon$. So, the next step consists of determining eventual intersections of sequences $(\pm u_m)$ and $(\pm u'_n)$ for $m,n \geq 1$.

5 Proof of the main Theorem for $|c| \geq 159\,108$

In this section we apply the *congruence method* introduced in [4] to obtain lower bound for $|U|$. Combining that result with a generalization of *Bennett's theorem*, we are able to solve the system (20) and (21) for large values of $|c|$.

5.1 A lower bound for a solution

Definition 5 *Let $a,b,d \in \mathbb{Z}_M$ and $d \neq 0$. We say that a is* congruent b modulo d *if there exists $x \in \mathbb{Z}_M$ such that $a - b = dx$. We write $a \equiv b \pmod{d}$.*

Lemma 6 *Let $|c| \geq 2$. Sequences (u_m) and (u'_n) given by (26) and (27) satisfy the following inequalities*

$$(2|c|-3)^m \leq |u_m| \leq (2|c|+3)^m,\ (2|c|-3)^n \leq |u'_n| \leq (2|c|+3)^n, \tag{28}$$

for $m,n \geq 0$.

Proof. The inequality for $|u'_n|$ is given in [12, Lemma 5.5.]. Similarly, we prove the other one. First, we show by induction that $(|u_m|)$ is a growing sequence. Evidently,

$$|u_1| = |2c+1| \geq 2|c| - 1 \geq 1 = |u_0|.$$

If $|u_m| \geq |u_{m-1}|$ for some $m \in \mathbb{N}$, then

$$|u_{m+1}| \geq |2c+2||u_m| - |u_{m-1}| \geq |2c+2||u_m| - |u_m| = (2|c|-3)|u_m| \geq |u_m|.$$

Since, $|u_0| = (2|c|-3)^0$ and $|u_1| \geq 2|c| - 1 \geq (2|c|-3)^1$, the previous inequality $|u_{m+1}| \geq (2|c|-3)|u_m|$ also implies that $|u_m| \geq (2|c|-3)^m$ for $m \geq 0$. Also, since

$$|u_0| = (2|c|+3)^0, \ \ |u_1| = |2c+1| \leq (2|c|+1) \leq (2|c|+3)^1.$$

and

$$|u_{m+1}| \leq (2|c|+2)|u_m| + |u_{m-1}| \leq (2|c|+3)|u_m|,$$

we obtain that $|u_m| \leq (2|c|+3)^m$ for $m \geq 0$. ■

Lemma 7 *Sequences* $(\pm u_m)$ *and* $(\pm u'_n)$ *given by (26) and (27) satisfy the following congruences*

$$u_m \equiv \varepsilon(1 + m(m+1)c) \pmod{4c^2}, \tag{29}$$
$$u'_n \equiv (-1)^n \varepsilon(1 - n(n+1)c) \pmod{4c^2}, \tag{30}$$

for $m, n \geq 0$.

Proof. The congruence relation for u'_n has already been proved in [12, Lemma 6.2.]. The other relation can be easily shown by induction. Recall that $u_0 = \varepsilon$ and $u_1 = \varepsilon(2c+1)$. Hence, (29) is true for $m = 0, 1$. Now, assume that $u_k \equiv \varepsilon(1 + k(k+1)c) \pmod{4c^2}$, for $k < m$ and $m \geq 2$. We obtain

$$\begin{aligned} u_m &= (2c+2)u_{m-1} - u_{m-2} \equiv (2c+2)\varepsilon(1 + (m-1)mc) - \varepsilon(1 + (m-2)(m-1)c) \\ &\equiv \varepsilon(1 + 2m(m-1)c^2 + m(m+1)c) \equiv \varepsilon(1 + m(m+1)c) \pmod{4c^2}, \end{aligned}$$

for $m \geq 2$. ■

Proposition 8 *Let* $c \notin S_c$. *If* $u_m = \pm u'_n$, *then*

$$m \geq \sqrt{2|c| + 0.25} - 0.5 \quad or \quad n \geq \sqrt{2|c| + 0.25} - 0.5 \quad or \quad m = n = 0.$$

Proof. If $u_m = \pm u'_n$, then Lemma 7 implies that

$$\varepsilon(1 + m(m+1)c) \equiv \pm(-1)^n \varepsilon(1 - n(n+1)c) \pmod{4c^2}.$$

Therefore we have the following congruence relation

$$\varepsilon(1 \mp (-1)^n) \equiv 0 \pmod{2c}.$$

If $\varepsilon(1 \mp (-1)^n) \neq 0$, then $|\varepsilon(1 \mp (-1)^n)| = 2$ and $|c| = 1$, which is not possible. So, we conclude that $\mp(-1)^n = -1$ and

$$\varepsilon(1 + m(m+1)c) \equiv \varepsilon(1 - n(n+1)c) \pmod{4c^2}.$$

Furthermore,

$$\varepsilon\left(\frac{m(m+1)}{2} + \frac{n(n+1)}{2}\right) \equiv 0 \pmod{2c}. \tag{31}$$

Consider the algebraic integer

$$A = \varepsilon\left(\frac{m(m+1)}{2} + \frac{n(n+1)}{2}\right).$$

It is clear that $A \neq 0$ for $m > 0$ or $n > 0$. So, (31) implies that $|A| \geq 2|c|$. Hence,

$$m(m+1) \geq 2|c| \ \text{ or } \ n(n+1) \geq 2|c|,$$

ie.

$$m \geq \sqrt{2|c| + 0.25} - 0.5 \ \text{ or } \ n \geq \sqrt{2|c| + 0.25} - 0.5.$$

■

Finally, the previous proposition yields a lower bound for a nontrivial solution of equations (20) and (21).

Corollary 9 *Let $c \notin S_c$. If $U \in \mathbb{Z}_M \backslash \{\pm\varepsilon\}$ is a solution of the system of equations (20) and (21), then*

$$|U| \geq (2|c| - 3)^{\sqrt{2|c|+0.25}-0.5}.$$

Proof. If follows straight away from Lemma 6 and Proposition 8. ■

5.2 An upper bound for a solution

The number of solutions of simultaneous Pellian equations can be bounded by using a theorem on simultaneous approximations by rationals to the square roots of rationals near 1 introduced by M. Bennett in [2]. In fact, we need its generalization for imaginary quadratic fields stated and proved in [12]. Namely, we use the following theorem:

Theorem 10 ([12, Theorem 7.1]) *Let $\theta_i = \sqrt{1 + \frac{a_i}{T}}$ for $1 \leq i \leq m$, with a_i pairwise distinct imaginary quadratic integers in $K := \mathbb{Q}(\sqrt{-D})$ with $0 < D \in \mathbb{Z}$ for $i = 0, \ldots, m$ and let T be an algebraic integer of K. Furthermore, let $A := \max |a_i|$, $|T| > A$ and $a_0 = 0$ and*

$$l = c_m \tfrac{(m+1)^{m+1}}{m^m} \cdot \tfrac{|T|}{|T|-A}, \qquad L = |T|^m \tfrac{(m+1)^{m+1}}{4m^m \prod_{0\leq i<j\leq m} |a_j - a_i|^2} \cdot \left(\tfrac{|T|-A}{|T|}\right)^m,$$

$$p = \sqrt{\tfrac{2|T|+3A}{2|T|-2A}}, \qquad P = |T| \cdot 2^{m+3} \tfrac{\prod_{0\leq i<j\leq m} |a_i - a_j|^2}{\min_{i\neq j} |a_i - a_j|^{m+1}} \cdot \tfrac{2|T|+3A}{2|T|},$$

where $c_m = \frac{3\Gamma\left(m-\frac{1}{2}\right)}{4\sqrt{\pi}\Gamma(m+1)}$, *such that* $L > 1$, *then*

$$\max\left(\left|\theta_1 - \frac{p_1}{q}\right|, \ldots, \left|\theta_m - \frac{p_m}{q}\right|\right) > cq^{-\lambda}$$

for all algebraic integers $p_1, \ldots, p_m, q \in K$, *where*

$$\lambda = 1 + \frac{\log P}{\log L} \quad \textit{and}$$
$$C^{-1} = 2mpP\left(\max\{1, 2l\}\right)^{\lambda - 1}.$$

The first step in the application of Theorem 10 consists of choosing suitable values for θ_1 and θ_2. Let $(U, V, Z) \in \mathbb{Z}_M^3$ be a solution of system of Pellian equations (20) and (21). The candidates for θ_1 and θ_2 are

$$\theta_1^{(1)} = \pm\sqrt{\frac{c+2}{c}},\ \theta_2^{(1)} = \pm\sqrt{\frac{c-2}{c}},\ \theta_1^{(2)} = -\theta_1^{(1)},\ \theta_2^{(2)} = -\theta_2^{(1)}, \tag{32}$$

where the signs are chosen such that

$$|V - \theta_1^{(1)}U| < |V - \theta_1^{(2)}U| \ \text{ and } \quad |Z - \theta_2^{(1)}U| < |V - \theta_2^{(2)}U|.$$

The next lemma shows that $\frac{V}{U}$ and $\frac{Z}{U}$ are good approximations to the algebraic numbers $\theta_1^{(1)}$ and $\theta_2^{(1)}$.

Lemma 11 *Let* $|c| > 2$. *If* $(U, V, Z) \in \mathbb{Z}_M^3$ *is a solution of (20) and (21), then*

$$\max\left\{\left|\theta_1^{(1)} - \frac{V}{U}\right|, \left|\theta_2^{(1)} - \frac{Z}{U}\right|\right\} \leq \frac{2}{\sqrt{|c|(|c|-2)}}|U|^{-2}.$$

Proof. We have

$$\begin{aligned} |V - \theta_1^{(2)}U| &\geq \frac{1}{2}(|V - \theta_1^{(1)}U| + |V - \theta_1^{(2)}U|) \\ &\geq \frac{1}{2}|U||\theta_1^{(1)} - \theta_1^{(2)}| = |U|\left|\sqrt{\frac{c+2}{c}}\right| \geq |U|\sqrt{\frac{|c|-2}{|c|}}, \end{aligned}$$

This implies

$$\left|\theta_1^{(1)} - \frac{V}{U}\right| = \left|\frac{c+2}{c} - \frac{V^2}{U^2}\right|\left|\theta_1^{(2)} - \frac{V}{U}\right|^{-1} \leq \frac{2}{|c||U|^2}\sqrt{\frac{|c|}{|c|-2}} = \frac{2}{\sqrt{|c|(|c|-2)}}|U|^{-2}.$$

Inequality

$$\left|\theta_2^{(1)} - \frac{Z}{U}\right| \leq \frac{2}{\sqrt{|c|(|c|-2)}}|U|^{-2}$$

is proved in [12, Lemma 8.1]. ■

The inputs of Theorem 10 are $m = 2$, $\theta_1 = \theta_1^{(1)}$, $\theta_2 = \theta_2^{(1)}$, $a_1 = 2$, $a_2 = -2$, $A = 2$, $T = c$, with $|T| = |c| > 2$,

$$
\begin{aligned}
l &= \frac{27}{64}\frac{|c|}{|c|-2}, \quad L = \frac{27}{4096}(|c|-2)^2 > 1 \text{ if } \ |c| \geq 15,\\
p &= \sqrt{\frac{|c|+3}{|c|-2}}, \quad P = 1024(|c|+3),\\
\lambda &= 1 + \frac{\log 1024 + \log(|c|+3)}{\log 27 - \log 4096 + 2\log(|c|-2)},\\
C^{-1} &= 4pP(\max\{1, \frac{27}{32}\frac{|c|}{|c|-2}\})^{\lambda-1} = 4pP = 4096(|c|+3)\sqrt{\frac{|c|+3}{|c|-2}} \text{ if } \ |c| \geq 13.
\end{aligned}
$$

Finally, Lemma 11 and Theorem 10 for $p_1 = V$, $p_2 = Z$ and $q = U$ give us the following inequality

$$\frac{2}{|U|^2\sqrt{|c|(|c|-2)}} \geq \max\left\{\left|\theta_1 - \frac{V}{U}\right|, \left|\theta_2 - \frac{Z}{U}\right|\right\} > C|U|^{-\lambda},$$

ie.

$$\frac{2C^{-1}}{\sqrt{|c|(|c|-2)}} > |U|^{2-\lambda}.$$

So, if $2 - \lambda > 0$, then the obtained upper bound for $|U|$ is

$$\log|U| < \frac{\log(\frac{2C^{-1}}{\sqrt{|c|(|c|-2)}})}{2-\lambda}. \tag{33}$$

We now examine the condition $f(|c|) = 2 - \lambda > 0$, where

$$f(t) = 1 - \frac{\log 1024 + \log(t+3)}{\log 27 - \log 4096 + 2\log(t-2)}, \ t > 2.$$

For $t > 15$, $f(t)$ is a strictly growing function and since $\lim_{t\to\infty} f(t) = \frac{1}{2}$, there exists t_0 such that $f(t) > 0$ for $t \geq t_0$. Since, $f(155\,352) > 0$ and $f(155\,351) < 0$, we conclude that the condition $2 - \lambda > 0$ is fulfilled for $|c| \geq 155\,352$. Now, we use the lower bound for $|U|$ given in Corollary 9 and obtain

$$\log|U| \geq (\sqrt{2|c| + 0.25} - 0.5)\log(2|c|-3), \ |c| > 2. \tag{34}$$

Comparing (33) and (34) we get the inequality

$$(\sqrt{2|c|+0.25} - 0.5)\log(2|c|-3) < \frac{\log(8192 \cdot \frac{(|c|+3)}{\sqrt{|c|(|c|-2)}}\sqrt{\frac{|c|+3}{|c|-2}})}{1 - \frac{\log 1024(|c|+3)}{\log\frac{27}{4096}(|c|-2)^2}},$$

which does not hold for $|c| \geq 159\,108$. Therefore, we have proved the following assertion.

Proposition 12 *For* $|c| \geq 159\,108$*, the only solutions of the system (20) and (21) are* $(U, V, Z) = (\pm\varepsilon, \pm\varepsilon, \pm\varepsilon)$ *with mix signs and* $\varepsilon = 1,\ i,\ \omega,\ \omega^2$ *corresponding to* $\mu = 1, -1, \omega,$ ω^2*, respectively.*

Let $(p, q) \in \mathbb{Z}_M^2$ be a solution of (19) and let $|c| \geq 159\,108$. From Proposition 12 and equations in (22), we have

$$U = p^2 + q^2 = \pm\varepsilon, \quad V = p^2 + 2pq - q^2 = \pm\varepsilon, \quad Z = -p^2 + 2pq + q^2 = \pm\varepsilon,$$

where $\varepsilon = 1,\ i,\ \omega,\ \omega^2$. Adding V and Z yields $2pq = 0, \pm\varepsilon$. Since $|2pq| \geq 2$ or $|2pq| = 0$, we have $2pq = 0$. Hence, either p or q is equal to 0 which implies $p^4 = \mu$ and $q = 0$ or $q^4 = \mu$ and $p = 0$, where $\mu \in \{1, -1, \omega, \omega^2\}$. Therefore the following theorem follows immediately.

Theorem 13 *Let* $c \notin S_c$*. If the equation (19) is solvable in* $(p, q) \in \mathbb{Z}_M^2$*, then* $\mu \in \{1, -1, \omega, \omega^2\}$ *where* $\omega = \frac{1}{2}(-1 + \sqrt{-3})$*. Furthermore, if* $|c| \geq 159\,108$*, then all solutions of (19) are given by*

1. $(p, q) \in \{(0, \pm 1), (\pm 1, 0), (0, \pm i), (\pm i, 0)\} \cap \mathbb{Z}_M^2$ *if* $\mu = 1$*;*
2. $(p, q) \in \{(0, \pm\omega), (\pm\omega, 0)\} \cap \mathbb{Z}_M^2$ *if* $\mu = \omega$*;*
3. $(p, q) \in \{(0, \pm\omega^2), (\pm\omega^2, 0)\} \cap \mathbb{Z}_M^2$ *if* $\mu = \omega^2$*.*

Note, that if $\mu = -1$ and $|c| \geq 159\,108$, then there is no solution of (19). Equations in (18) and Theorem 13 imply the following proposition right away.

Proposition 14 *If* $|c| \geq 159\,108$*, then all non-equivalent generators of power integral basis of* $\mathcal{O} = \mathbb{Z}_M[\xi]$ *over* $\mathbb{Z}_M$ *are* $\alpha = \xi,\ 2\xi - 2c\xi^2 + \xi^3$*.*

Remark 15 *Note that* $(U, V, Z) = (\pm\varepsilon, \pm\varepsilon, \pm\varepsilon)$ *with mix signs and* $\varepsilon = 1,\ i,\ \omega,\ \omega^2$ *corresponding to* $\mu = 1, -1, \omega,\ \omega^2$*, respectively, are solutions of the system (20) and (21) for all* $c \in \mathbb{Z}_M$*. This implies that* $\alpha = \xi,\ 2\xi - 2c\xi^2 + \xi^3$ *are non-equivalent generators of power integral basis for all* $c \in \mathbb{Z}_M \setminus \{0, \pm 2\}$*.*

6 Applying Baker's theory for $|c| < 159\,108$

The famous theorem of Baker and Wüstholz from [1] says:

Theorem 16 *If* $\Lambda = b_1\alpha_1 + \cdots + b_l\alpha_l \neq 0$*, where* $\alpha_1, \ldots, \alpha_l$ *are algebraic numbers and* $b_1, \ldots, b_l$ *are rational integers, then*

$$\log|\Lambda| \geq -18(l+1)!\,l^{l+1}(32d)^{l+2}h(\alpha_1)\cdots h(\alpha_l)\log(2ld)\log B,$$

where $B = \max\{|(b_1|, \ldots, |b_l|\}$*,* d *is the degree of the number field generated by* $\alpha_1, \ldots, \alpha_l$ *over the rationals* $\mathbb{Q}$*,*

$$h'(\alpha) = \max\{h(\alpha), \frac{1}{d}|\log\alpha|, \frac{1}{d}\},$$

and $h(\alpha)$ *denotes the standard logarithmic Weil height.*

The previous theorem can be applied on finding intersections of binary recursive sequences close to sequences of the form $\alpha\beta^m$, where α, β are algebraic integers. So, first we make sure that our sequences u_m and u'_n are close to that form.

Assume that $|c| < 159\,108$, $c \notin S_c$ and $\mathrm{Re}(c) \geq 0$. Indeed, if $\mathrm{Re}(c) < 0$, then by replacing c in the system of equations (20), (21) by $-c$, we obtain the system

$$(c-2)U^2 - cV^2 = -2\mu,\ cZ^2 - (c+2)U^2 = -2\mu,$$

which corresponds to the initial system (20), (21) by switching places of Z and V. Let us agree that the square root of a complex number $z = re^{i\varphi}$, $-\pi < \varphi \leq \pi$ is given by

$$\sqrt{z} = \sqrt{r}e^{i\frac{\varphi}{2}},$$

ie. the one with a positive real part (or the principal square root).

Let (U, V, Z) be solution of the system (20) and (21). In Section 4 we showed that than there exists $m \geq 0$ such that $U = \pm u_m$, where the sequence (u_m) is given by (26). Solving the recursion in (26) yields an explicit expression for u_m:

$$u_m = \varepsilon\frac{1}{2\sqrt{c(c+2)}}\Big((c+\sqrt{c(c+2)})(c+1+\sqrt{c(c+2)})^m - (c-\sqrt{c(c+2)})(c+1-\sqrt{c(c+2)})^m\Big). \tag{35}$$

Since $Re(c) \geq 0$, $|c+1+\sqrt{c(c+2)}|\cdot|c+1-\sqrt{c(c+2)}| = 1$ and $|c+1+\sqrt{c(c+2)}| \neq |c+1-\sqrt{c(c+2)}|$ for $c \neq 0, -1, -2$, we have $|c+1+\sqrt{c(c+2)}| > 1$ (and $|c+1-\sqrt{c(c+2)}| < 1$). So, we put

$$P = \frac{1}{\sqrt{c+2}}(c+\sqrt{c(c+2)})(c+1+\sqrt{c(c+2)})^m. \tag{36}$$

Through algebraic manipulation it can be shown

$$u_m = \frac{\varepsilon}{2\sqrt{c}}(P + \frac{2c}{c+1}P^{-1}), \tag{37}$$

since $\sqrt{c(c+2)} = \sqrt{c}\sqrt{(c+2)}$ if $\mathrm{Re}(c) \geq 0$ (because $\sqrt{z_1 z_2} = \sqrt{z_1}\sqrt{z_2}$ is not true in general) and

$$P^{-1} = \frac{\sqrt{c+2}}{2c}(\sqrt{c(c+2)} - c)(c+1-\sqrt{c(c+2)})^m.$$

Analogously, there exists $n \geq 0$ such that $U = \pm u'_n$, where the sequence (u'_n) is given by (27) and it's explicit expression is

$$u'_n = \varepsilon\frac{1}{2\sqrt{c(c-2)}}\Big((c+\sqrt{c(c-2)})(c-1+\sqrt{c(c-2)})^n - (c-\sqrt{c(c-2)})(c-1-\sqrt{c(c-2)})^n\Big). \tag{38}$$

Also, since $|c-1+\sqrt{c(c-2)}| \neq |c-1-\sqrt{c(c-2)}|$ for $c \neq 0, 1, 2$ and $|c-1+\sqrt{c(c-2)}| \cdot |c-1-\sqrt{c(c-2)}| = 1$, we put

$$Q = \frac{1}{\sqrt{c-2}}(c+\sqrt{c(c-2)})(c-1+\sqrt{c(c-2)})^n, \tag{39}$$

if $|c-1+\sqrt{c(c-2)}|>1$. Alternatively, if $|c-1+\sqrt{c(c-2)}|<1$, ie. $|c-1-\sqrt{c(c-2)}|>1$, we put

$$Q=\frac{1}{\sqrt{c-2}}(c-\sqrt{c(c-2)})(c-1-\sqrt{c(c-2)})^n. \tag{40}$$

To be more precise, if $\mathrm{Re}(c)>1$ or $\mathrm{Re}(c)=1$ and $\mathrm{Im}(c)>0$, then Q is given by (39) and also $\sqrt{c(c-2)}=\sqrt{c}\sqrt{c-2}$. In the other hand, if $0\le \mathrm{Re}(c)<1$ or $\mathrm{Re}(c)=1$ and $\mathrm{Im}(c)<0$, then $\sqrt{c(c-2)}=-\sqrt{c}\sqrt{c-2}$ and Q is defined by (40). Note that, in both cases, Q can be given by

$$Q=\frac{1}{\sqrt{c-2}}(c+\sqrt{c}\sqrt{c-2})(c-1+\sqrt{c}\sqrt{c-2})^n. \tag{41}$$

Similarly to the previous case, we have

$$u_n'=\pm\frac{\varepsilon}{2\sqrt{c}}(Q-\frac{2c}{c-2}Q^{-1}), \tag{42}$$

where

$$Q^{-1}=\frac{\sqrt{c-2}}{2c}(c-\sqrt{c}\sqrt{c-2})(c-1-\sqrt{c}\sqrt{c-2})^n.$$

The theorem of Baker and Wüstholz (Theorem 16) will be applied on the form

$$\Lambda=\log\frac{|Q|}{|P|}.$$

6.1 Estimates on $|\Lambda|$

First, we have to estimate the lower bounds for $|P|$ and $|Q|$. Since $|c|\neq 1$, then $|c|\ge\sqrt{2}$ and

$$|P|\ge 11.6, \tag{43}$$

for $m\ge 2$. Indeed, the inequality (43) follows from the fact that

$$\left|\frac{c+\sqrt{c(c+2)}}{\sqrt{c+2}}\right|=|\sqrt{c}|\left|\frac{\sqrt{c}}{\sqrt{c+2}}+1\right|\ge 1\cdot 1=1,$$

since $|\sqrt{c}|=\sqrt{|c|}\ge 1$, $\mathrm{Re}\left(\frac{\sqrt{c}}{\sqrt{c+2}}\right)\ge 0$ and

$$\left|c+1+\sqrt{c(c+2)}\right|^2\ge(|c|+2)^2\ge(\sqrt{2}+2)^2\ge 11.6.$$

Similarly, if $|c|\ge\sqrt{2}$ and $n\ge 2$, then (41) implies

$$|Q|\ge|\sqrt{c}|\left|\frac{\sqrt{c}}{\sqrt{c-2}}+1\right|\left|c-1+\sqrt{c}\sqrt{c-2}\right|^2\ge\left(\sqrt{2}+1\right)^2>5.8. \tag{44}$$

In the case $|c|\ge 2$, (43) and (44) can be immediately improved to

$$|P|\ge 16,\ \ |Q|\ge 9.$$

Since there are finitely many integers c such that $\sqrt{2} \le |c| < 2$, we can easily obtain lower bounds for $|P|$ and $|Q|$ assuming $|c| \ge \sqrt{2}$. Indeed,

$$|P| \ge \min\{16, \min\{P_c : c \in T\}\}, \ |Q| \ge \min\{9, \min\{Q_c : c \in T\}\}$$

where

$$P_c = \left|\sqrt{\frac{c}{c+2}}\right| \left|\sqrt{c} + \sqrt{c+2}\right| \left|c + 1 + \sqrt{c(c+2)}\right|^2,$$

$$Q_c = \left|\sqrt{\frac{c}{c-2}}\right| \left|\sqrt{c} \pm \sqrt{c-2}\right| \left|c - 1 + \sqrt{c}\sqrt{c-2}\right|^2,$$

and

$$T = \{\pm\sqrt{-2}, \frac{1 \pm \sqrt{-7}}{2}, \pm\sqrt{-3}, \frac{3 \pm \sqrt{-3}}{2}, \frac{1 \pm \sqrt{-11}}{2}\}, \tag{45}$$

ie. T is the set of all integers in $\mathbb{Q}(\sqrt{-D})$ such that $\sqrt{2} \le |c| < 2$, $c \notin S_c$ and $\mathrm{Re}(c) \ge 0$. Finally, we get

$$|P| \ge 16, |Q| \ge 9,$$

for $|c| \ge \sqrt{2}$ and $n, m \ge 2$ (because $\min\{P_c : c \in T\} \ge 16$ and $\min\{Q_c : c \in T\} \ge 9$). Using these bounds, we have to show that the value of $\left|\log \frac{|Q|}{|P|}\right|$ is small enough. Assuming that

$$u_m = \pm u'_n, \ m \ge 2, n \ge 2,$$

relations (37) and (42) imply

$$P \pm Q = \pm \frac{2c}{c-2} Q^{-1} + \frac{2c}{c+2} P^{-1}. \tag{46}$$

So,

$$\begin{aligned} ||P| - |Q|| \le |P \mp Q| \ &\le \ \left|\frac{2c}{c-2}\right| |Q|^{-1} + \left|\frac{2c}{c+2}\right| |P|^{-1} \\ &< \ 2 \cdot 5 \cdot \frac{1}{9} + 2 \cdot 1 \cdot \frac{1}{16} = 1.24. \end{aligned} \tag{47}$$

(Note that $\left|\frac{c}{c-2}\right| \le 5$ for $c \in \mathbb{Z}_M$, $c \ne 2$, $\mathrm{Re}(c) \ge 0$, and $\left|\frac{c}{c+2}\right| \le 1$ for $c \in \mathbb{Z}_M$ and $\mathrm{Re}(c) \ge 0$.) Since,

$$\frac{||P| - |Q||}{|P|} < 1.24|P|^{-1} < 1,$$

we have

$$\left|\log \frac{|Q|}{|P|}\right| = \log \left|1 - \frac{|P| - |Q|}{|P|}\right| \le \frac{||P| - |Q||}{|P|} + \left(\frac{||P| - |Q||}{|P|}\right)^2.$$

Also, the inequality $||P| - |Q|| < 1.24$ implies that

$$|P| < |Q| + 1.24 \le |Q| + 1.24\frac{|Q|}{9} < 1.14|Q|,$$

or equivalently

$$|Q|^{-1} < 1.14|P|^{-1}.$$

By putting that into (47), we get

$$||P|-|Q|| < \left|\frac{2c}{c-2}\right| \cdot 1.14|P|^{-1} + \left|\frac{2c}{c+2}\right| |P|^{-1} < 13.4|P|^{-1}.$$

Finally,

$$\left|\log\frac{|Q|}{|P|}\right| < 13.4|P|^{-2} + (13.4|P|^{-2})^2 \le (13.4 + (13.4\frac{1}{16})^2)|P|^{-2} < 14.11|P|^{-2}.$$

Furthermore, it can be shown that

$$|\Lambda| = \left|\log\frac{|Q|}{|P|}\right| < 14.11|P|^{-2} < 14.11 \cdot \left(\sqrt{2}+2\right)^{-2m} < 3^{-m},$$

for $|c| \ge \sqrt{2}$ and $m, n \ge 2$. Similarly we find

$$\frac{||P|-|Q||}{|Q|} < 1.24|Q|^{-1} < 1$$

and

$$\begin{aligned} |\Lambda| &= \left|\log\frac{|P|}{|Q|}\right| = \log\left|1 - \frac{|P|-|Q|}{|Q|}\right| \le \frac{||P|-|Q||}{|Q|} + \left(\frac{||P|-|Q||}{|Q|}\right)^2 \\ &< 14|Q|^{-2} < 14 \cdot \left(\sqrt{2}+1\right)^{-2n} < (1.55)^{-n}. \end{aligned}$$

It remains to show that there are no solutions of $u_m = \pm u'_n$ in cases when $m = 1$ or $n = 1$. For $c \in \mathbb{Z}_M$, $|c| \ge \sqrt{2}$ and $\mathrm{Re}(c) \ge 0$, we have

$$|u_1| = |2c+1| \le 2|c|+1, \quad |u'_1| = |2c-1| \le 2|c|+1$$

and

$$|u_m| > (2\sqrt{1+|c|^2}-1)^m, \quad |u'_n| > (2\sqrt{1+|c|^2}-1)^{n-1}(2|c|-1),$$

for $m, n \ge 2$ (where last inequalities are obtained similarly to those in Lemma 6). Since,

$$|u_1| \le 2|c|+1 < (2\sqrt{1+|c|^2}-1)(2|c|-1) \le |u'_n|, \text{ if } n \ge 2$$

$$|u'_1| \le 2|c|+1 < (2\sqrt{1+|c|^2}-1)^2 \le |u_m|, \text{ if } m \ge 2$$

we conclude that the equations $u_1 = \pm u'_n$ and $u'_1 = \pm u_m$ have no solution for $m, n \ge 2$, i.e. for $m, n \ge 1$ (because $u_1 \ne \pm u'_1$).

6.2 The condition $\Lambda \ne 0$

Without proving that $\Lambda \ne 0$, i.e. $|P| \ne |Q|$, we cannot apply Theorem 16. This proof is rather complicated and involves several cases.

First, we show that $P \ne \pm Q$. Let us assume that $P = \pm Q$. According to (46), the following possibilities may occur:

$$\frac{c}{c^2-4} = 0,$$

which is obviously not possible ($c \neq 0, \pm 2$), and

$$P^2 = \frac{2c}{c^2-4}.$$

In Section 6.1, we have shown that $|P|^2 \geq 16^2$. Since

$$\left|\frac{2c}{c^2-4}\right| \leq \frac{2|c|}{||c|^2-4|} \leq \frac{2\cdot\sqrt{5}}{5-4} < 5,$$

for $|c| \geq \sqrt{5}$ and

$$\left|\frac{2c}{c^2-4}\right| \leq \max\{\left|\frac{2c}{c^2-4}\right| \;:\; c \in T_1\} < 1,\;\; c \in T_1,$$

where $T_1 = \{c \in \mathbb{Z}_M \backslash S_c : \sqrt{2} \leq |c| < \sqrt{5}\}$, ie.

$$T_1 = T \cup \left\{\pm 2\sqrt{-1}, \frac{3\pm\sqrt{-7}}{2}, \frac{1\pm\sqrt{-15}}{2}\right\}$$

and T is given in (45), we obtain a contradiction.

Before presenting other cases, let us take a closer look at $|P|$ and $|Q|$ from an algebraic point of view. According to (36), we have

$$\frac{P}{\sqrt{c}} = \frac{c+\sqrt{c(c+2)}}{\sqrt{c(c+2)}}(c+1+\sqrt{c(c+2)})^m = a + b\alpha = a + \frac{b_1}{c+2}\alpha, \tag{48}$$

where $\alpha = \sqrt{c(c+2)}$ and $a, b_1 \in \mathbb{Z}_M$. Similarly, (39) and (40) imply that

$$\frac{Q}{\sqrt{c}} = d + e\beta = d + \frac{e_1}{c-2}\beta, \tag{49}$$

where $\beta = \sqrt{c(c-2)}$ and $d, e_1 \in \mathbb{Z}_M$. It follows straight away that

$$\begin{aligned} u_m &= \frac{\varepsilon}{2}(a+b\alpha+a-b\alpha) = \varepsilon a, \\ u'_n &= \frac{\varepsilon}{2}(d+e\beta+d-e\beta) = \varepsilon d, \end{aligned}$$

where we have used the explicit expressions (35) and (38) for u_m and u'_n. Since $u_m = \pm u'_n$, we get

$$a = \pm d.$$

Note that $a \neq 0$, $d \neq 0$, because $|u_m|, |u'_n| > 0$ for $m, n \geq 2$. We have

$$\left|\frac{P}{\sqrt{c}}\right|^2 = (a+b\alpha)(\overline{a}+\overline{b}\overline{\alpha}) = |a|^2 + (\overline{a}b)\alpha + (a\overline{b})\overline{\alpha} + |b|^2|\alpha|^2$$

and analogously

$$\left|\frac{Q}{\sqrt{c}}\right|^2 = (d+e\alpha)(\overline{d}+\overline{e}\overline{\beta}) = |d|^2 + (\overline{d}e)\beta + (d\overline{e})\overline{\beta} + |e|^2|\beta|^2.$$

If the algebraic extension $\mathbb{Q}(\sqrt{-D})(\alpha, \overline{\alpha}, \beta, \overline{\beta})$ is considered as a vector space over $\mathbb{Q}(\sqrt{-D})$, then $\{1, \alpha, \overline{\alpha}, |\alpha|^2, \beta, \overline{\beta}, |\beta|^2\}$ is its generating set and $\left|\frac{P}{\sqrt{c}}\right|^2$ and $\left|\frac{Q}{\sqrt{c}}\right|^2$ are the elements of the vector subspaces $span(1, \alpha, \overline{\alpha}, |\alpha|^2)$ and $span(1, \beta, \overline{\beta}, |\beta|^2)$, respectively. Before continuing with the proof, we establish the following useful claims:

Lemma 17 *If $c \notin \{0, \pm 1, \pm 2\}$, then $\alpha, \beta \notin \mathbb{Q}(\sqrt{-D})$.*

Proof. Indeed, we can show that $\alpha \in \mathbb{Q}(\sqrt{-D})$ if and only if $c = 0, -1, -2$. Let $\alpha \in \mathbb{Q}(\sqrt{-D})$. Note that $\alpha \in \mathbb{Q}(\sqrt{-D})$ if and only if $c(c+2) = t^2$ for some $t \in \mathbb{Z}_M$. Therefore, $c = -1 \pm \sqrt{t^2+1}$, where $t^2 + 1 = s^2$ for some $s \in \mathbb{Z}_M$. Note that $t, s, c \in \mathbb{Z}_M$ and $t \pm s$ are units in $\mathbb{Z}_M$. It is easy to check that only possibilities are $c = 0, -1, -2$. It can be proved similarly that $\beta \in \mathbb{Q}(\sqrt{-D})$ if and only if $c = 0, 1, 2$. ■

Lemma 18 *If B_1 is a basis of the subspace $span(1, \alpha, \overline{\alpha}, |\alpha|^2)$, then $B_1 = \{1, \alpha, \overline{\alpha}, |\alpha|^2\}$ or $B_1 = \{1, \alpha\}$. Set $\{1, \alpha\}$ is a basis of $span(1, \alpha, \overline{\alpha}, |\alpha|^2)$ if and only if $\overline{\alpha} = K\alpha$, $K \in \mathbb{Q}(\sqrt{-D})$. The analogous statement is true for a basis of $span(1, \beta, \overline{\beta}, |\beta|^2)$.*

Proof. According to Lemma 17, it is obvious that $\{1, \alpha\}$ is a linearly independent set. Let $\overline{\alpha} = A + C\alpha$, for $A, C \in \mathbb{Q}(\sqrt{-D})$. By squaring it, we obtain $2AC\alpha = \overline{\alpha}^2 - A^2 - \alpha^2 C^2 \in \mathbb{Q}(\sqrt{-D})$. Since $\alpha \notin \mathbb{Q}(\sqrt{-D})$, we have that $AC = 0$. If $C = 0$, then $\overline{\alpha} = A \in \mathbb{Q}(\sqrt{-D})$, a contradiction. If $A = 0$, then $\overline{\alpha} = C\alpha$ and $|\alpha|^2 = C\alpha^2 \in \mathbb{Q}(\sqrt{-D})$ which imply $B_1 = \{1, \alpha\}$.

If $\{1, \alpha, \overline{\alpha}\}$ is a linearly independent set and $|\alpha|^2 = A + C\alpha + E\overline{\alpha}$ for $A, C, E \in \mathbb{Q}(\sqrt{-D})$, then by squaring it we get

$$\underbrace{|\alpha|^4}_{\in \mathbb{Q}(\sqrt{-D})} = \underbrace{A^2 + C^2\alpha^2 + E^2\overline{\alpha}^2}_{\in \mathbb{Q}(\sqrt{-D})} + 2AC\alpha + 2AE\overline{\alpha} + 2CE\underbrace{(A + C\alpha + E\overline{\alpha})}_{|\alpha|^2},$$

a linear combination of $1, \alpha, \overline{\alpha}$. So, $C(A + E) = 0$ and $E(A + C) = 0$. If $A = C = 0$, then $|\alpha|^2 = E\overline{\alpha}$ implies $\alpha \in \mathbb{Q}(\sqrt{-D})$, a contradiction. Also, other two cases end with a contradiction ($A = E = 0$ implies $\overline{\alpha} \in \mathbb{Q}(\sqrt{-D})$ and $C = E = 0$ implies that $\{\alpha, \overline{\alpha}\}$ is a linearly dependent set). ■

Lemma 19 *Let $c \notin \{0, \pm 1, \pm 2\}$. If $\beta \in span(1, \alpha, \overline{\alpha}, |\alpha|^2)$, then $B_1 = \{1, \alpha, \overline{\alpha}, |\alpha|^2\}$ is a basis of the subspace $span(1, \alpha, \overline{\alpha}, |\alpha|^2)$ and $\beta = K\overline{\alpha}$ or $\beta = K\,|\alpha|^2$, for some $K \in \mathbb{Q}(\sqrt{-D})$. The analogous statement is true if $\alpha \in span(1, \beta, \overline{\beta}, |\beta|^2)$.*

Proof. Let $\beta \in span(1, \alpha, \overline{\alpha}, |\alpha|^2)$. Obviously, this implies $\overline{\beta}$, $|\beta|^2 \in span(1, \alpha, \overline{\alpha}, |\alpha|^2)$, too. If we assume that $\beta = K\alpha$ for some $K \in \mathbb{Q}(\sqrt{-D})$, then $K = \pm\frac{\sqrt{c^2-4}}{c-2}$. Therefore, $c^2 - 4 = r^2$ for some $r \in \mathbb{Z}_M$. Since $c, r \in \mathbb{Z}_M$ and $|c \pm r| \leq 4$, by checking all possibilities, we find $c = 0, \pm 1, -2$. (Similarly, if $\beta = K\alpha$ for $K \in \mathbb{Q}(\sqrt{-D})$, then $c = 0, \pm 1, 2$).

If $B_1 = \{1, \alpha\}$ is basis of subspace $span(1, \alpha, \overline{\alpha}, |\alpha|^2)$, then $\beta = L + K\alpha$ some $L, K \in \mathbb{Q}(\sqrt{-D})$. Then, by squaring it, it is easy to see $\beta = L \in \mathbb{Q}(\sqrt{-D})$ or $\beta = K\alpha$, which is impossible.

If $B_1 = \{1, \alpha, \overline{\alpha}, |\alpha|^2\}$ is basis of subspace $span(1, \alpha, \overline{\alpha}, |\alpha|^2)$, then $\beta = L + K\alpha + K'\overline{\alpha} + K''\,|\alpha|^2$ for some $L, K, K', K'' \in \mathbb{Q}(\sqrt{-D})$. Similarly as before, we obtain $\beta = L \in \mathbb{Q}(\sqrt{-D})$

or $\beta = K\alpha$ or $\beta = K'\overline{\alpha}$ or $\beta = K'' |\alpha|^2$. Therefore, we might have $\beta = K'\overline{\alpha}$ or $\beta = K'' |\alpha|^2$, since first two cases are impossible. ■

Furthermore,

$$\left|\frac{P}{\sqrt{c}}\right|^2 - \left|\frac{Q}{\sqrt{c}}\right|^2 \in V = span(1, \alpha, \overline{\alpha}, |\alpha|^2, \beta, \overline{\beta}, |\beta|^2).$$

There are several possibilities for choosing a basis B for V from its generating set $\{1, \alpha, \overline{\alpha}, |\alpha|^2, \beta, \overline{\beta}, |\beta|^2\}$:

(a) $B = \{1\}$. This happens if and only if $\alpha, \beta \in \mathbb{Q}(\sqrt{-D})$. So, this is not possible according to Lemma 17.

(b) $B = \{1, \alpha\}$ (or $B = \{1, \beta\}$). This is also not possible. Indeed, in this case $B_1 = \{1, \alpha\}$ is basis of subspace $span(1, \alpha, \overline{\alpha}, |\alpha|^2)$ and $\beta \in span(1, \alpha, \overline{\alpha}, |\alpha|^2)$, which contradicts Lemma 19.

(c) $B = \{1, \alpha, \overline{\alpha}, |\alpha|^2\}$ (or $B = \{1, \beta, \overline{\beta}, |\beta|^2\}$). In this case $B_1 = \{1, \alpha, \overline{\alpha}, |\alpha|^2\}$ is basis of subspace $span(1, \alpha, \overline{\alpha}, |\alpha|^2)$ and $\beta \in span(1, \alpha, \overline{\alpha}, |\alpha|^2)$. This case implies that $\beta = K\overline{\alpha}$ or $\beta = K|\alpha|^2$ for $K \in \mathbb{Q}(\sqrt{-D})$ according to Lemma 19.

(d) $B = \{1, \alpha, \beta\}$. This implies that $\overline{\alpha} = K\alpha$ and $\overline{\beta} = L\beta$, for $K, L \in \mathbb{Q}(\sqrt{-D})$ according to Lemma 18.

(e) $B = \{1, \alpha, \beta, \overline{\beta}, |\beta|^2\}$ (or $B = \{1, \alpha, \overline{\alpha}, |\alpha|^2, \beta\}$). Here, we have $\overline{\alpha} = K\alpha$ for $K \in \mathbb{Q}(\sqrt{-D})$ (or $\overline{\beta} = K\beta$ for $K \in \mathbb{Q}(\sqrt{-D})$)

(f) $B = \{1, \alpha, \overline{\alpha}, |\alpha|^2, \beta, \overline{\beta}, |\beta|^2\}$.

In what follows, we show that $|P| \neq |Q|$ in each of possible cases (c) to (f) unless $\mathrm{Re}(c) = 0$. Assume that $|P| = |Q|$, ie.

$$0 = \left|\frac{P}{\sqrt{c}}\right|^2 - \left|\frac{Q}{\sqrt{c}}\right|^2 = (\overline{a}b)\alpha + (a\overline{b})\overline{\alpha} + |b|^2|\alpha|^2 - (\overline{d}e)\beta - (d\overline{e})\overline{\beta} - |e|^2|\beta|^2. \qquad (50)$$

Case (f): Let $B = \{1, \alpha, \overline{\alpha}, |\alpha|^2, \beta, \overline{\beta}, |\beta|^2\}$ basis B for V. Since the set $\{\alpha, \overline{\alpha}, |\alpha|^2, \beta, \overline{\beta}, |\beta|^2\}$ is linearly independent, all coefficients have to be zero:

$$\overline{a}b = a\overline{b} = |b|^2 = \overline{d}e = d\overline{e} = |e|^2 = 0.$$

This implies $b = e = 0$ and

$$P = a\sqrt{c} = \pm d\sqrt{c} = \pm Q,$$

which is not possible.

Case (e): The assumption is that the set $B = \{1, \alpha, \beta, \overline{\beta}, |\beta|^2\}$ form a basis for V. In this case we know that $\overline{\alpha} = K\alpha$ and $|\alpha|^2 = K\alpha^2$ for $K \in \mathbb{Q}(\sqrt{-D})$. Obviously, $K \neq 0$. So, (50) imply

$$(|b|^2K\alpha^2)1 + (\overline{a}b + a\overline{b}K)\alpha - (\overline{d}e)\beta - (d\overline{e})\overline{\beta} - |e|^2|\beta|^2 = 0.$$

The coefficients must be zero:

$$|b|^2 \underbrace{K\alpha^2}_{\neq 0} = \overline{a}b + a\overline{b}K = \overline{d}e = d\overline{e} = |e|^2 = 0.$$

Hence, $b = e = 0$ and

$$P = a\sqrt{c} = \pm d\sqrt{c} = \pm Q,$$

which is not possible. Similarly, we obtain a contradiction, if we assume $B = \{1, \alpha, \overline{\alpha}, |\alpha|^2, \beta\}$ is a basis for V.

Case (d): The set $B = \{1, \alpha, \beta\}$ form a basis for V. This is a situation when

$$\overline{\alpha} = K\alpha, \ |\alpha|^2 = K\alpha^2, \ \overline{\beta} = L\beta, \ |\beta|^2 = L\beta^2,$$

for $K, L \in \mathbb{Q}(\sqrt{-D})$ and $K, L \neq 0$. Furthermore, K, L are units in $\mathbb{Q}(\sqrt{-D})$, ie. $|K| = |L| = 1$. Implementing that into (50) we get

$$(|b|^2 K\alpha^2 - |e|^2 L\beta^2)1 + (\overline{a}b + a\overline{b}K)\alpha - (\overline{d}e + d\overline{e}L)\beta = 0,$$

and

$$|b|^2 K\alpha^2 = |e|^2 L\beta^2, \ \overline{a}b = -a\overline{b}K, \ \overline{d}e = -d\overline{e}L.$$

Assume that $b, e \neq 0$. Since $a, d \neq 0$, substituting

$$K = -\frac{\overline{a}b}{a\overline{b}}, \ L = -\frac{\overline{d}e}{d\overline{e}},$$

we get

$$|b|^2 \frac{\overline{a}b}{a\overline{b}} \alpha^2 = |e|^2 \frac{\overline{d}e}{d\overline{e}} \beta^2 \ \Leftrightarrow \ \left(\frac{\overline{a}b}{|a|}\right)^2 \alpha^2 = \left(\frac{\overline{d}e}{|d|}\right)^2 \beta^2.$$

Also since, $a = \pm d$, we have

$$b^2\alpha^2 = e^2\beta^2.$$

So,

$$(b\alpha - e\beta)(b\alpha + e\beta) = 0$$

and this leads to $b = e = 0$, because α, β are linearly independent, which gives again $P = \pm Q$. A contradiction!

Case (c): Recall that $\{1, \alpha, \overline{\alpha}, |\alpha|^2\}$ forms a basis of V and $\beta = K\overline{\alpha}$ or $\beta = K|\alpha|^2$ for $K \in \mathbb{Q}(\sqrt{-D})$.

If $\beta = K|\alpha|^2$, then $\overline{\beta} = \overline{K}|\alpha|^2$ and $|\beta|^2 = |K|^2|\alpha|^4 \in \mathbb{Q}(\sqrt{-D})$. So, (50) implies that

$$(-|e|^2|K|^2|\alpha|^4) \cdot 1 + (\overline{a}b)\alpha + (a\overline{b})\overline{\alpha} + (|b|^2 - \overline{d}eK - d\overline{e}\overline{K})|\alpha|^2 = 0.$$

Therefore,

$$|e|^2|K|^2|\alpha|^4 = 0, \ \overline{a}b = 0, \ |b|^2 - \overline{d}eK - d\overline{e}\overline{K} = 0.$$

Evidently $e = b = 0$ which imply $P = \pm Q$, a contradiction.

If $\beta = K\overline{\alpha}$, then $\overline{\beta} = \overline{K}\alpha$ and $|\beta|^2 = |K|^2|\alpha|^2$. Notice that $\overline{\beta} \neq L\beta$ for all $L \in \mathbb{Q}(\sqrt{-D})$. (If $\overline{\beta} = L\beta$, then $\beta = L^{-1}K\alpha$, which is not possible by Lemma 19.) According to (50) we have

$$(\overline{a}b - d\overline{e}\overline{K})\alpha + (a\overline{b} - \overline{d}eK)\overline{\alpha} + (|b|^2 - |e|^2|K|^2)|\alpha|^2 = 0,$$

and

$$a\overline{b} - \overline{d}eK = 0, \ |b|^2 - |e|^2|K|^2 = 0.$$

Therefore,

$$K = \frac{a\overline{b}}{\overline{d}e} = \pm\frac{a\overline{b}}{\overline{a}e}. \tag{51}$$

From (48) and (49) we obtain

$$a^2 - b^2c\,(c+2) = \frac{2}{c+2}, \qquad a^2 - e^2c\,(c-2) = -\frac{2}{c-2} \tag{52}$$

and

$$cb_1^2 - (c+2)\,a^2 = -2, \qquad (c-2)\,a^2 - ce_1^2 = -2, \tag{53}$$

which again implies

$$e^2c\,(c-2) - b^2c\,(c+2) = \frac{4c}{c^2-4} \ \text{ and } \ (c+2)\,e_1^2 - (c-2)\,b_1^2 = 4. \tag{54}$$

Equation (51) implies $|e\beta| = |b\alpha|$, which again implies $\left|e_1^2\,(c+2)\right| = \left|b_1^2\,(c-2)\right|$. Let

$$X = (c+2)\,e_1^2 \quad \text{and} \quad Y = (c-2)\,b_1^2.$$

Therefore, we have $X - Y = 4$ and $|X| = |Y|$, which implies $\operatorname{Re} X = 2$, $\operatorname{Re} Y = -2$. On the other hand, from (51) and (54) we obtain

$$\frac{c^2-4}{c\cdot\overline{a}^2}\left(\overline{b}^2a^2\overline{\alpha}^2 - \overline{a}^2b^2\alpha^2\right) = 4. \tag{55}$$

Since $\overline{b}^2a^2\overline{\alpha}^2 - \overline{a}^2b^2\alpha^2 = 2\operatorname{Im}\left(a\overline{b}\overline{\alpha}\right)^2 i$, equation (55) implies

$$\operatorname{Re}\left(\frac{c^2-4}{c\overline{a}^2}\right) = 0. \tag{56}$$

The condition (56) is equivalent to the condition $\operatorname{Re}\left(\frac{(c^2-4)a^2}{c}\right) = 0$. On the other hand, from (53) and $\operatorname{Re}\left((c+2)\,e_1^2\right) = 2$, we obtain

$$\begin{aligned}\operatorname{Re}\left(\frac{(c^2-4)\,a^2}{c}\right) &= \operatorname{Re}\left(\frac{(c+2)\,(ce_1^2-2)}{c}\right)\\ &= \operatorname{Re}\left(-2 - \frac{4}{c} + e_1^2\,(c+2)\right) = -\operatorname{Re}\left(\frac{4}{c}\right) = 0,\end{aligned}$$

which again implies $\operatorname{Re} c = 0$, ie. $c = vi$, $v \in \mathbb{Z}\left(\sqrt{D}\right)$, $v \neq 0, \pm1$. In general we have $\overline{\sqrt{z}} = \sqrt{\overline{z}}$, if $z \in \mathbb{C}\backslash\mathbb{R}^-$ and $\overline{\sqrt{z}} = -\sqrt{\overline{z}}$, if $z \in \mathbb{R}^-$. Since, we have $\beta = \sqrt{vi\,(vi-2)}$,

$\alpha = \sqrt{vi\,(vi+2)}$ and $\overline{(vi-2)\,vi} = vi\,(vi+2) \notin \mathbb{R}^-$, then $\beta = \overline{\alpha}$ ie. $K=1$. Also, from (56) we obtain that $\operatorname{Re}\left(\frac{-iv}{v^2+4}\,(\overline{a})^2\right) = 0$, which again implies $\overline{a} = -a$ or $\overline{a} = a$. Therefore, since $K=1$, from (51) we have $e = \pm\overline{b}$ and distinguish four cases:

1. If $\overline{a} = a = d$, then $e = \frac{a\overline{b}}{d} = \overline{b}$ and $\frac{P}{\sqrt{c}} = a + b\alpha$, $\frac{Q}{\sqrt{c}} = a + \overline{b}\overline{\alpha}$;
2. $\overline{a} = -a$, $a = d$, then $e = \frac{a\overline{b}}{d} = -\overline{b}$ and $\frac{P}{\sqrt{c}} = a + b\alpha$, $\frac{Q}{\sqrt{c}} = a - \overline{b}\overline{\alpha}$;
3. $\overline{a} = a$, $a = -d$, then $e = \frac{a\overline{b}}{d} = -\overline{b}$ and $\frac{P}{\sqrt{c}} = a + b\alpha$, $\frac{Q}{\sqrt{c}} = -a - \overline{b}\overline{\alpha}$;
4. $\overline{a} = -a$, $a = -d$, then $e = \frac{a\overline{b}}{d} = \overline{b}$ and $\frac{P}{\sqrt{c}} = a + b\alpha$, $\frac{Q}{\sqrt{c}} = -a + \overline{b}\overline{\alpha}$.

Note that each of the above cases implies that $|P| = |Q|$ and therefore $\Lambda = 0$. In what follows we show that in this particular case the equation $u_m = \pm u'_n$, $m, n > 0$, has no solution.

First we will show that $\overline{a} = \pm a$ imply $b\alpha \neq \pm\overline{b}\overline{\alpha}$. It is enough to show $\operatorname{Im}(b\alpha)^2 \neq 0$. Suppose $\operatorname{Im}(b\alpha)^2 = 0$. Then, from (52) we have

$$(b\alpha)^2 = b^2 c\,(c+2) = a^2 - \frac{2}{vi+2}.$$

Since, we have $\operatorname{Im}(b\alpha)^2 = \operatorname{Im} a^2 = 0$, then $\operatorname{Im}\frac{2}{vi+2} = 0$ which again implies $v = 0$, a contradiction.

From (46) we obtain

$$\frac{P}{\sqrt{c}} - \frac{Q}{\sqrt{c}} = -\frac{2}{(c-2)}\cdot\frac{\sqrt{c}}{Q} + \frac{2}{(c+2)}\cdot\frac{\sqrt{c}}{P}, \quad \text{if} \quad a = d, \tag{57}$$

$$\frac{P}{\sqrt{c}} + \frac{Q}{\sqrt{c}} = \frac{2}{(c-2)}\cdot\frac{\sqrt{c}}{Q} + \frac{2}{(c+2)}\cdot\frac{\sqrt{c}}{P}, \quad \text{if} \quad a = -d. \tag{58}$$

If $\overline{a} = a = d$, then (57) imply

$$b\alpha - \overline{b}\overline{\alpha} = \frac{2}{2-vi}\cdot\frac{1}{\overline{a}+\overline{b}\overline{\alpha}} + \frac{2}{2+vi}\cdot\frac{1}{a+b\alpha}.$$

Since $\operatorname{Re}\left(b\alpha - \overline{b}\overline{\alpha}\right) = 0$ and $\operatorname{Im}\left(\frac{2}{2-vi}\cdot\frac{1}{\overline{a}+\overline{b}\overline{\alpha}} + \frac{2}{2+vi}\cdot\frac{1}{a+b\alpha}\right) = 0$, we obtain $b\alpha = \overline{b}\overline{\alpha}$, a contradiction. Similarly, we obtain contradiction in other three cases. Analogous results are obtained if $B = \{1, \beta, \overline{\beta}, |\beta|^2\}$ is a basis for V.

Note that if $c = vi$, $v \in \mathbb{Z}[\sqrt{D}]$, $v \neq 0, \pm1$, then $\beta = \overline{\alpha}$ and, according to Lemma 19, $\{1, \alpha, \overline{\alpha}, |\alpha|^2\}$ forms a basis of $V = \mathbb{Q}(\sqrt{-D})(\alpha, \overline{\alpha}, \beta, \overline{\beta})$, that is Case (c). Therefore we have proved the following assertion.

Proposition 20 *Let $c \notin S_c$ and $\Lambda = \log\frac{|P|}{|Q|}$. Then*

i) $\Lambda \neq 0$ *if and only if* $\operatorname{Re}(c) \neq 0$.

ii) *If* $\operatorname{Re}(c) = 0$, *then the equation* $u_m = \pm u'_n$ *has no solution for* $m, n > 0$.

Corollary 21 *If $c \neq \pm\sqrt{-1}$ and $\operatorname{Re}(c) = 0$, then all non-equivalent generators of power integral basis of $\mathcal{O} = \mathbb{Z}_M[\xi]$ over $\mathbb{Z}_M$ are $\alpha = \xi$, $2\xi - 2c\xi^2 + \xi^3$.*

6.3 A reduction procedure

We are now ready to apply Theorem 16 to our linear form in logarithms of algebraic numbers

$$\Lambda = \log|Q| - \log|P| = n\log\eta - m\log\vartheta + \log\xi,$$

where

$$\eta = |c - 1 + \sqrt{c}\sqrt{c-2}|,\ \vartheta = |c + 1 + \sqrt{c(c+2)}|,\ \xi = \left|\frac{\sqrt{c+2}(\sqrt{c}+\sqrt{c-2})}{\sqrt{c-2}(\sqrt{c}+\sqrt{c+2})}\right|.$$

First we have to calculate the standard logarithmic Weil height of η, ϑ and ξ. Since, the standard logarithmic Weil height $h(\alpha)$ is bounded by

$$h(\alpha) \leq \frac{1}{k}\log\left(a_0\prod_{i=1}^{k}\max\{1, |\alpha^{(i)}|\}\right),$$

where the algebraic number α is a root of $a_0\prod_{i=1}^{k}(x-\alpha^{(i)})$. Note that η, ϑ and ξ are roots of the following polynomials

$$p_1(x) = 1 - 4(1 - c - 4\overline{c} + c\overline{c})x^2 + (6 - 8c + 4c^2 - 8\overline{c} + 4\overline{c}^2)x^4 - 4(1 - c - 4\overline{c} + c\overline{c})x^6 + x^8,$$

$$p_2(x) = 1 - 4(1 + c + \overline{c} + c\overline{c})x^2 + (6 + 8c + 4c^2 + 8\overline{c} + 4\overline{c}^2)x^4 - 4(1 + c + \overline{c} + c\overline{c})x^6 + x^8,$$

$$\begin{aligned}p_3(x) = {} & \tfrac{(-2+c)^8(-2+\overline{c}^8)}{(2+c)^8(2+\overline{c})^8} - 4\tfrac{(-2+c)^8(-2+\overline{c}^8)}{(2+c)^7(2+\overline{c})^7}x^2 - 24\tfrac{(-2+c)^7(-2+\overline{c}^7(-5+c^2+\overline{c}^2)}{(2+c)^7(2+\overline{c})^7}x^4 \\
& +4\tfrac{(-2+c)^7(-2+\overline{c})^7(-35+4c^2+4\overline{c}^2)}{(2+c)^6(2+\overline{c})^6}x^6 + 4\tfrac{(-2+c)^6(-2+\overline{c})^6(455-116c^2+4c^4-116\overline{c}^2+44c^2\overline{c}^2+4\overline{c}^4)}{(2+c)^6(2+\overline{c})^6}x^8 \\
& -4\tfrac{(-2+c)^6(-2+\overline{c})^6(273-36c^2-36\overline{c}^2+16c^2\overline{c}^2)}{(2+c)^5(2+\overline{c})^5}x^{10} \\
& -8\tfrac{(-2+c)^5(-2+\overline{c})^5(-1001+253c^2-8c^4+253\overline{c}^2-72c^2\overline{c}^2+8c^4\overline{c}^2-8\overline{c}^4+8c^2\overline{c}^4)}{(2+c)^5(2+\overline{c})^5}x^{12} \\
& +4\tfrac{(-2+c)^5(-2+\overline{c})^5(-715+88c^2+88\overline{c}^2+16c^2\overline{c}^2)}{(2+c)^4(2+\overline{c})^4}x^{14} \\
& 2\tfrac{(-2+c)^4(-2+\overline{c})^4(6435-1584c^2+48c^4-1584\overline{c}^2+16c^2\overline{c}^2+64c^4\overline{c}^2+48\overline{c}^4+64c^2\overline{c}^4)}{(2+c)^4(2+\overline{c})^4}x^{16} \\
& +4\tfrac{(-2+c)^4(-2+\overline{c})^4(-715+88c^2+88\overline{c}^2+16c^2\overline{c}^2)}{(2+c)^3(2+\overline{c})^3}x^{18} \\
& -8\tfrac{(-2+c)^3(-2+\overline{c})^3(-1001+253c^2-8c^4+253\overline{c}^2-72c^2\overline{c}^2+8c^4\overline{c}^2-8\overline{c}^4+8c^2\overline{c}^4)}{(2+c)^3(2+\overline{c})^3}x^{20} \\
& -4\tfrac{(-2+c)^3(-2+\overline{c})^3(273-36c^2-36\overline{c}^2+16c^2\overline{c}^2)}{(2+c)^2(2+\overline{c})^2}x^{22} + 4\tfrac{(-2+c)^2(-2+\overline{c})^2(455-116c^2+4c^4-116\overline{c}^2+44c^2\overline{c}^2+4\overline{c}^4)}{(2+c)^2(2+\overline{c})^2}x^{24} \\
& +4\tfrac{(-2+c)^2(-2+\overline{c})^2(-35+4c^2+4\overline{c}^2)}{(2+c)(2+\overline{c})}x^{26} - 24\tfrac{(-2+c)(-2+\overline{c})(-5+c^2+\overline{c}^2)}{(2+c)(2+\overline{c})}x^{28} - 4(-2+c)(-2+\overline{c})x^{30} + x^{32},\end{aligned}$$

respectively.

Each conjugate of an algebraic number in absolute value can be bounded by $|\alpha'| \leq \max\{1, k|a'|\}$, where $|a'| = \max\{|a_0|, \ldots, |a_{k-1}|\}$ and $a_0, \ldots, a_{k-1}$ are coefficients of the related monic polynomial $\prod_{i=1}^{k}(x-\alpha^{(i)})$. Hence,

$$h(\alpha) \leq \log(\max\{1, k|a'|\}).$$

It is easy to see, that each coefficient of the polynomials $p_1(x)$ and $p_2(x)$ can be bounded (in the absolute value) by $6 + 16|c| + 8|c|^2$. All coefficients of $p_3(x)$ can be bounded by $(|c|+2)^{16}(6435 + 3168|c|^2 + 112|c|^4 + 128|c|^6)$ - a very rough bound. So,

$$h(\eta),\ h(\vartheta) \leq \log(8(6 + 16|c| + 8|c|^2)) < 28.12,$$

$$h(\xi) \leq \log\left(32(|c|+2)^{16}(6435 + 3168|c|^2 + 112|c|^4 + 128|c|^6)\right) < 271.82,$$

and obviously $h'(\eta)$, $h'(\vartheta)$ and $h'(\xi)$ are less than the values given above. Finally, since $d \leq 32 \cdot 8 \cdot 8$ we have

$$-\log|\Lambda| \leq 18 \cdot 4! \cdot 3^4 (32 \cdot 2048)^5 28.12^2 \cdot 271.82 \cdot \log(2 \cdot 3 \cdot 2048) \log l < 8.6 \cdot 10^{34} \log l,$$

where $l = \max\{m, n\}$. If $l = m$, applying $|\Lambda| < 3^{-m}$ to the previous inequality, we get

$$\frac{m}{\log m} < 7.8 \cdot 10^{34}$$

which does not hold for $m \geq 6.7 \cdot 10^{36}$. Therefore, we solve

$$|\Lambda| = |\log \eta| \left| n - m \frac{\log \vartheta}{\log \eta} + \frac{\log \xi}{\log \eta} \right| < 3^{-m}, \ m < 6.7 \cdot 10^{36},$$

ie.

$$|m\theta - n + \gamma| < \delta \cdot 3^{-m} \tag{59}$$

where $\theta = \frac{\log \vartheta}{\log \eta}$, $\gamma = -\frac{\log \xi}{\log \eta}$ and $\delta = \frac{1}{|\log \eta|}$.

If $l = n$, applying $|\Lambda| < 1.55^{-n}$ to the previous inequality, we get

$$\frac{n}{\log n} < 2 \cdot 10^{35}$$

which does not hold for $n \geq 1.715 \cdot 10^{37}$. Therefore, we solve

$$|\Lambda| = |\log \vartheta| \left| m - n \frac{\log \eta}{\log \vartheta} + \frac{\log \xi}{\log \vartheta} \right| < (1.55)^{-n}, \quad n < 1.715 \cdot 10^{37}$$

ie.

$$|n\theta' - n + \gamma'| < \delta' \cdot 1.55^{-n} \tag{60}$$

where $\theta' = \frac{\log \eta}{\log \vartheta}$, $\gamma' = \frac{\log \xi}{\log \vartheta}$ and $\delta' = \frac{1}{|\log \vartheta|}$.

Now we will apply the reduction method similar to one described in [3].

Lemma 22 ([3, Lemma 4a]) *Let M be a positive integer and let p/q be a convergent of the continued fraction expansion of θ such that $q > 6M$. Furthermore, let $\varepsilon = \|\gamma q\| - M\|\theta q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then the inequality*

$$|m\theta - n + \gamma| < \delta a^{-m}$$

has no integer solutions m and n such that $\log(\delta q/\varepsilon)/\log a \leq m \leq M$.

Since our bound for the absolute value of c is very huge (almost 160 000), we perform reductions only for $|c| \leq 1000$, $c \in \mathbb{Z}_M$. We obtained that (59) and (60) has no integer solutions for $m \geq n > 31$ and $n \geq m > 67$, respectively. The reason for not achieving a better bound for m and n is because θ and θ' are very close to 1 and hence their first convergent is too large, although for certain values of c the reduction procedure is very efficient. For an impression, $c = 1 + 984\sqrt{-1}$ with related $\theta' \approx 1.000000272$ and $q_1 = 3\,672\,014$ (the denominator of the first convergent) represents a non-efficient example of reduction

($m \leq n \leq 67$), while on $c = 10 + \sqrt{-61}$ with related $\theta \approx 1.039$ the reduction works much better ($n \leq m \leq 2$). Finally, we showed that the equations $u_m = \pm u'_n$ for $1 \leq m, n \leq 67$ have no solutions in $\mathbb{Z}_M$ except $c = \pm 1, \pm 2$. (Note that according to (26) and (26), u_k and u'_k are k-th degree polynomials in the variable c. So, solving $u_m = \pm u'_n$ reduces to finding roots of certain polynomials in $\mathbb{Z}_M$.)

Computational aspects. All reductions and calculations were performed in Wolfram's Mathematica 9.0 with 150-digit precision. Since the algorithm for $|c| \leq 200$, $|c| \leq 400$ and $|c| \leq 1000$ took respectively 1718s, 9757s and 99710s, we estimate that the time required to do all computations for $|c| < 159108$ is more then 10^{10} sec.

Therefore, we have proved:

Proposition 23 *If $|c| \leq 1000$ and $c \notin S_c$, then all non-equivalent generators of power integral basis of $\mathcal{O} = \mathbb{Z}_M[\xi]$ over $\mathbb{Z}_M$ are $\alpha = \xi$, $2\xi - 2c\xi^2 + \xi^3$.*

7 On the case $c \in S_c$

So far, we have observed the case when the parameter $c \notin S_c$, where the set S_c is given by (3). Note that if $c \in S_c$, then on at least one of the equation of the system (20) and (21) we can not apply Lemma 4. Indeed, in these cases, there are additional classes of solutions of the equation (20) or (21), or there exists only finitely many solutions of those equations. Also note, if $(p,q) = (a,b)$ is a solution of the Thue equation (19) for $c = c_0$, then $(p,q) = (b,a)$ is a solution of this equation for $c = -c_0$. Therefore, it is enough to observe only c's from the set S_c with $Re(c) \geq 0$. Furthermore, all $c \in S_c$ are from only one imaginary quadratic field except $c = \pm 1$ that belong to each field $M = \mathbb{Q}\left(\sqrt{-D}\right)$. Thus, for each $c \in S_c$, $c \neq \pm 1$, we have to find additional classes of solutions of the equation (20) or (21) (see [5]) and repeat the entire procedure from previous sections. This situation is much simpler because we have a specific value of c and each c is from exactly one field. On the other hand, we need to find intersections of at least four recursive series.

For $c = 1$ Thue equation (19) have the form

$$p^4 - 2p^3q + 2p^2q^2 + 2pq^3 + q^4 = \mu, \tag{61}$$

and the related system is

$$V^2 - 3U^2 = -2\mu, \quad U^2 + Z^2 = 2\mu.$$

By Lemma 4, solutions of the first equation are $(V,U) = (\pm v_m, \pm u_m)$, where sequences (v_m) and (u_m) are given by

$$\begin{aligned} v_0 &= \varepsilon, & v_1 &= 5\varepsilon, & v_{m+2} &= 4v_{m+1} - v_m, \ \ m \geq 0, \\ u_0 &= \varepsilon, & u_1 &= 3\varepsilon, & u_{m+2} &= 4u_{m+1} - u_m, \ \ m \geq 0, \end{aligned}$$

where $\varepsilon = 1$, i, ω, ω^2 corresponds to $\mu = 1, -1, \omega, \omega^2$, respectively. Therefore, we have to observe the equation

$$U^2 + Z^2 = 2\mu, \tag{62}$$

for $\mu \in \left\{1, \omega, \omega^2\right\} \cap \mathbb{Q}\left(\sqrt{-D}\right)$, if $D \neq 1$ and $\mu \in \{1, -1\}$ if $D = 1$.

If $D = 1$, then -1 is a square in $\mathbb{Z}_M$, and the left side of (62) can be factorized as

$$U^2 + Z^2 = U^2 - (-1)\, Z^2 = (U - iZ)\,(U + iZ) = 2\mu, \quad \mu = 1, -1.$$

This implies that the equation (62) has only finitely many solutions

$$(U, Z) = (\pm 1, \pm 1)\,, \text{ if } \ \mu = 1 \ \text{ and } \quad (U, Z) = (\pm i, \pm i)\,, \text{ if } \mu = -1,$$

which again implies that all solutions of system of relative Pellian equations are given by

$$\begin{aligned} (U, V, Z) &= (\pm 1, \pm 1, \pm 1)\,, \text{ if } \ \mu = 1 \\ (U, V, Z) &= (\pm i, \pm i, \pm i)\,, \text{ if } \ \mu = -1. \end{aligned}$$

Hence, if $\mu = 1$, then the solutions of the corresponding Thue equation are

$$(p, q) \in \{(0, \pm 1), (\pm 1, 0), (0, \pm i), (\pm i, 0)\}$$

and if $\mu = -1$, then there are no solutions.

Note, that for $c = 1$ the corresponding Thue equation (61) can be transformed into equation

$$X^2 + 3Y^2 = \mu \tag{63}$$

by putting $X = \pm\left(p^2 - pq - q^2\right)$ and $Y = \pm pq$. The equation (63) has infinitely many solutions in all rings $\mathbb{Z}_M$, except in the ring of integers of the field $\mathbb{Q}\left(\sqrt{-3}\right)$ because -3 is a square in that ring. In that case equation (63) can be factorized as

$$X^2 + 3Y^2 = \left(X - \sqrt{-3}Y\right)\left(X + \sqrt{-3}Y\right) = \mu,$$

where $\mu = 1, \omega, \omega^2$. This implies that the equation (63) has only finitely many solutions

$$(X, Y) \in \left\{(\pm 1, 0)\,, (\pm \omega, 0)\,, \left(\pm \omega^2, 0\right)\right\}.$$

Since $Y = \pm pq = 0$ for each solution from above, we conclude that all solutions Thue equation (61) are

$$\begin{aligned} (p, q) &\in \{(0, \pm 1), (\pm 1, 0)\}, \quad \text{if } \ \mu = 1, \\ (p, q) &\in \{(0, \pm \omega), (\pm \omega, 0)\}, \quad \text{if } \ \mu = \omega, \\ (p, q) &\in \{(0, \pm \omega^2), (\pm \omega^2, 0)\}, \quad \text{if } \ \mu = \omega^2. \end{aligned}$$

In the ring of integers $\mathbb{Z}_M$ of the field $M = \mathbb{Q}\left(\sqrt{-D}\right)$, where $D \neq 1, 3$ for $c = 1$, we have to find all solutions of the equation

$$U^2 + Z^2 = 2. \tag{64}$$

In this case the equation (64) has infinitely many solutions and the form of these solutions depend on D.

Therefore, we have proved:

Proposition 24 *Let $M = \mathbb{Q}\left(\sqrt{-D}\right)$, where $D = 1, 3$. If $c = 1$ or $c = -1$, then non-equivalent generators of power integral basis of $\mathcal{O} = \mathbb{Z}_M[\xi]$ over $\mathbb{Z}_M$ are given by $\alpha = \xi$, $2\xi - 2\xi^2 + \xi^3$ or $\alpha = \xi$, $2\xi + 2\xi^2 + \xi^3$, respectively.*

8 On elements with the absolute index 1

Let $\mathbb{Q} \subset M \subset K$ be number fields with $m = [M : \mathbb{Q}]$ and $k = [K : M]$. Let $\mathcal{O}$ be either the ring of integers $\mathbb{Z}_K$ of K or an order of $\mathbb{Z}_K$. Denote $D_{\mathcal{O}}$ and D_M the discriminant of $\mathcal{O}$ and subfield M, respectively. Also, denote by $\gamma^{(i)}$ the conjugates of any $\gamma \in M$ $(i = 1, \ldots, m)$. Let $\delta^{(i,j)}$ be the images of $\delta \in K$ under the automorphisms of K leaving the conjugate field $M^{(i)}$ elementwise fixed $(j = 1, \ldots, k)$.

According to [11] for any primitive element $\alpha \in \mathcal{O}$ we have

$$I_{\mathcal{O}}(\alpha) = \left[\mathcal{O}^+ : \mathbb{Z}[\alpha]^+\right] = \left[\mathcal{O}^+ : \mathbb{Z}_M[\alpha]^+\right] \cdot \left[\mathbb{Z}_M[\alpha]^+ : \mathbb{Z}[\alpha]^+\right]. \tag{65}$$

The first factor we call the *relative index* of α and we have

$$I_{\mathcal{O}/M}(\alpha) = \left[\mathcal{O}^+ : \mathbb{Z}_M[\alpha]^+\right] =$$

$$= \frac{1}{\sqrt{|N_{M/\mathbb{Q}}(D_{\mathcal{O}/M})|}} \cdot \prod_{i=1}^{m} \prod_{1 \le j_1 < j_2 \le k} \left|\alpha^{(i,j_1)} - \alpha^{(i,j_2)}\right| \tag{66}$$

where $D_{\mathcal{O}/M}$ is relative discriminant of $\mathcal{O}$ over M. For the second factor we have

$$J(\alpha) = \left[\mathbb{Z}_M[\alpha]^+ : \mathbb{Z}[\alpha]^+\right] =$$

$$= \frac{1}{\sqrt{|D_M|}^{[K:M]}} \cdot \prod_{1 \le i_1 < i_2 \le m} \prod_{j_1=1}^{k} \prod_{j_2=1}^{k} \left|\alpha^{(i_1,j_1)} - \alpha^{(i_2,j_2)}\right|. \tag{67}$$

Generators α_0 of relative power integral bases of $\mathcal{O}$ over M have relative index $I_{\mathcal{O}/M}(\alpha_0) = 1$. The elements

$$\alpha = A + \varepsilon \cdot \alpha_0, \tag{68}$$

(where ε is a unit in M and $A \in \mathbb{Z}_M$) have the same relative index, and are called *equivalent* with α_0 over M. Equivalently, all elements $\alpha \in \mathcal{O}$ generating a power integral basis of $\mathcal{O}$ (over $\mathbb{Q}$), that is having $I_{\mathcal{O}}(\alpha) = 1$, must be of the form (68), where α_0 has relative index $I_{\mathcal{O}/M}(\alpha_0) = 1$. In order that α generates a power integral basis of $\mathcal{O}$ we must also have $J(\alpha) = 1$. Therefore for each $\alpha_0 \in \mathcal{O}$ with relative index $I_{\mathcal{O}/M}(\alpha_0) = 1$, we have to determine the unit $\varepsilon \in M$ and $A \in \mathbb{Z}_M$ such that $J(\alpha) = 1$.

We consider the octic field $K_c = \mathbb{Q}(\xi)$, where ξ is a root of the polynomial $f(t) = t^4 - 2ct^3 + 2t^2 + 2ct + 1$, where $c \in \mathbb{Z}_M \setminus \{0, \pm 2\}$, $M = \mathbb{Q}(\sqrt{-D})$ and D is a squarefree positive integer. Therefore, $m = [M : \mathbb{Q}] = 2$ and K_c is an extension of M of degree $k = [K_c : M] = 4$.

We have proved that all generators of relative power integral bases of $\mathcal{O} = \mathbb{Z}_M[\xi]$ over M are given by

$$\alpha_1 = \xi, \quad \alpha_2 = 2\xi - 2c\xi^2 + \xi^3,$$

in the cases given in Theorem 1. Also, according to Remark 15, α_1 and α_2 are the generators of relative power integral bases for all $c \in \mathbb{Z}_M \setminus \{0, \pm 2\}$.

Proof of Theorem 3. Taking $\alpha_0 = \alpha_1, \alpha_2$ we calculate $J(\alpha)$ with the α in (68). For $-D \equiv 2, 3 \pmod 4$ an integral basis of M is given by $\{1, \vartheta\}$ with $\vartheta = \sqrt{-D}$. We have

$$\sqrt{|D_M|}^{[K:M]} = 16D^2.$$

We set $c = p + q\vartheta$ with integer parameters p, q. Let $A = a + b\vartheta$ with $a, b \in \mathbb{Z}$. Note that the product (67) in $J(\alpha)$ does not depend on a. We have $\varepsilon = \pm 1$ and for $-D = -1$ we also have $\varepsilon = \pm i$. The product

$$\prod_{j_1=1}^{4} \prod_{j_2=1}^{4} \left| \alpha^{(1,j_1)} - \alpha^{(2,j_2)} \right| \tag{69}$$

is of degree 16, depending on D, p, q and b. We calculated this product by Maple using symmetric polynomials. The result is a very complicated polynomial with integer coefficients of the above variables. We found that in each case the above product was divisible by $4096D^2$. Therefore dividing it by $16D^2$ the $J(\alpha)$ is divisible by 256. This implies that we cannot have $J(\alpha) = 1$, therefore we cannot have $I_{\mathcal{O}}(\alpha) = 1$. ■

Computational aspects

It was very difficult to perform the calculation of the product (69). We had to do it in several steps making simplifications by using symmetric polynomials in each step. Even so, this calculation has reached the limits of the capacities of Maple. We were not able to perform this calculation for $-D \equiv 1 \pmod 4$.

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