Establishing the minimal index in a parametric family of bicyclic biquadratic fields

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Abstract

Let $c \geq 3$ be positive integer such that c, $4c+1$, $c-1$ are square-free integers relatively prime in pairs. In this paper we find minimal index and determine all elements with minimal index in bicyclic biquadratic field $K = \mathbb{Q}\left(\sqrt{(4c+1)\,c}, \sqrt{(c-1)\,c}\right)$.

1 Introduction

Let K be an algebraic number field of degree n and $\mathcal{O}_K$ its ring of integers. For any $\alpha \in \mathcal{O}_K$

$$I(\alpha) = \left(\mathcal{O}_K^+ : \mathbb{Z}[\alpha]^+\right)$$

is the *index* of the element α, where $\mathcal{O}_K^+$ and $\mathbb{Z}[\alpha]^+$ respectively denote the additive groups of $\mathcal{O}_K$ and the polynomial ring $\mathbb{Z}[\alpha]$. If $K = \mathbb{Q}(\alpha)$ and $\alpha \in \mathcal{O}_K$, than we say that α is a primitive integer in the field K. The *minimal index* $\mu(K)$ of K is the minimum of the indices of all primitive integers in the field K. The greatest common divisor of indices of all primitive integers of K is called the *field index* of K, and will be denoted by $m(K)$. Therefore the minimal index $\mu(K)$ is divisible by the field index $m(K)$.

Let $\{1, \omega_2, ..., \omega_n\}$ be an integral basis of K. Let

$$L(\underline{X}) = X_1 + \omega_2 X_2 + ... + \omega_n X_n,$$

with conjugates $L_i(\underline{X}) = X_1 + \omega_2^{(i)} X_2 + ... + \omega_n^{(i)} X_n$, $i = 1, ..., n$. Then

$$D_{K/\mathbb{Q}}(L(\underline{X})) = \prod_{1 \leq i < j \leq n} \left(L_i(\underline{X}) - L_j(\underline{X})\right)^2$$

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is called discriminant of the linear form $L(\underline{X})$. We have

$$D_{K/\mathbb{Q}}(L(\underline{X})) = (I(X_2,...,X_n))^2 D_K,$$

where D_K denotes the discriminant of K and $I(X_2,...,X_n)$ is a homogenous polynomial in $n-1$ variables of degree $n(n-1)/2$ with rational integer coefficients which is called the *index form* corresponding to the integral basis $\{1,\omega_2,...,\omega_n\}$. It is well known that if the primitive integer $\alpha \in \mathcal{O}_K$ is represented in an integral basis as $\alpha = x_1 + x_2\omega_2 + ... + x_n\omega_n$, then the index of α is just $I(\alpha) = |I(x_2,...,x_n)|$.

If the number field K admits power integral basis $\{1,\alpha,...,\alpha^{n-1}\}$, i.e. if $\mathcal{O}_K = \mathbb{Z}[\alpha]$, it is called *monogenic.* Therefore, the element $\alpha \in \mathcal{O}_K$ generates a power integral basis if and only if $I(\alpha) = 1$. Consequently, number field K is monogenic if and only if $\mu(K) = 1$.

Biquadratic fields were considered by several authors. K. S. Williams [22] gave an explicit formula for integral basis and discriminant of these fields. T. Nakahara [20] proved that infinitely many fields of this type are monogenic, and on the other hand, for any given N there are infinitely many non monogenic fields of this type with minimal index $\mu(K) > N$. M. N. Gras, and F. Tanoe [17] established necessary and sufficient conditions for biquadratic fields being monogenic. I. Gaál, A. Pethő and M. Pohst [16] gave an algorithm for determining minimal index and all generators of integral bases in the totally real case by solving systems of simultaneous Pellian equations.

In the present paper we find the minimal index and determine all integral elements with minimal index in the family of totally real bicyclic biquadratic fields

$$K_c = \mathbb{Q}\left(\sqrt{(4c+1)c}, \sqrt{c(c-1)}\right) = \tag{1}$$
$$\mathbb{Q}\left(\sqrt{(4c+1)(c-1)}, \sqrt{c(c-1)}\right) = \mathbb{Q}\left(\sqrt{(4c+1)(c-1)}, \sqrt{(4c+1)c}\right).$$

We distinguish two cases according to c modulo 4. In both cases, by applying the method of [16]: first we reduced our problem to consider a family of systems of simultaneous Pellian equations. In order to find minimal index we use theory of continued fractions to determine all minimal values of the right hand side of the equations such that the system has solutions. In particular, we will use a characterization in terms of continued fractions of α of all fractions a/b satisfying the inequality

$$\left|\alpha - \frac{a}{b}\right| < \frac{M}{b^2},$$

where $M \in \mathbb{N}$, $M \leq 5$. After that finding all integral elements with minimal index reduces to solving the system of Pellian equations

$$(c-1)X^2 - cY^2 = -4, \tag{2}$$
$$(4c+1)X^2 - 4cT^2 = 4. \tag{3}$$

This system is very suitable for application of the method given in [7]. The main result of the present paper is the following theorem

Theorem 1 *Let $c \geq 3$ be a positive integer such that c, $4c+1$, $c-1$ are square-free integers relatively prime in pairs. Then K_c is totally real bicyclic biquadratic field and*

i) *its field index is $m(K_c) = 1$ for all c;*

ii) *the minimal index of K_c is: $\mu(K_c) = 5$ if $c = 3$; $\mu(K_c) = 40$ if $c \equiv 3 \pmod 4$, $c \geq 7$; $\mu(K_c) = 80$ if $c \equiv 2 \pmod 4$;*

iii) *all integral elements with minimal index are given by*

$$x_1 + x_2\sqrt{c(c-1)} + x_3\sqrt{c(4c+1)} + x_4\frac{\sqrt{c(c-1)} + \sqrt{(4c+1)(c-1)}}{2},$$

where $x_1 \in \mathbb{Z}$ and $(x_2, x_3, x_4) = \pm(-1, 0, 1), \pm(0, 0, 1)$ if $c = 3$; $(x_2, x_3, x_4) = \pm(1, \pm 1, 2), \pm(3, \pm 1, -2)$ if $c \equiv 3 \pmod 4$, $c \geq 7$; if $c \equiv 2 \pmod 4$, then all integral elements with minimal index are given by

$$x_1 + x_2\frac{1 + \sqrt{(4c+1)(c-1)}}{2} + x_3\sqrt{c(c-1)} + x_4\frac{\sqrt{c(c-1)} + \sqrt{(4c+1)c}}{2},$$

where $x_1 \in \mathbb{Z}$ and $(x_2, x_3, x_4) = \pm(\pm 2, 1, 2), \pm(\pm 2, 3, -2)$.

Note that c, $4c+1$, $c-1$ are integers relatively prime in pairs except when $c \equiv 1 (\bmod 5)$. Furthermore, by [10], there are infinitely many positive integers c for which $c(4c+1)(c-1)$ is square-free integer. Therefore, there are infinitely many positive integers c for which c, $4c+1$, $c-1$ are square-free integers relatively prime in pairs, which again implies that there are infinitely many totally real bicyclic biquadratic fields of the form (1).

2 Preliminaries

Let m, n denote distinct square-free integers. Let $l = \gcd(m, n)$ and let m_1, n_1 be defined by $m = lm_1$, $n = ln_1$. Under these conditions the quartic field $K = \mathbb{Q}(\sqrt{m}, \sqrt{n})$ has three distinct the quadratic subfields, namely $\mathbb{Q}(\sqrt{m})$, $\mathbb{Q}(\sqrt{n})$, $\mathbb{Q}(\sqrt{m_1 n_1})$ and Galois group V_4 (the Klein four group).

K.S. Williams [22] computed explicit formulae for integral basis and discriminant of the field $K = \mathbb{Q}(\sqrt{m}, \sqrt{n})$ in terms of m, n, m_1, n_1, l. He distinguished five cases according to the congruence behavior of m, n, m_1, n_1 modulo 4. In [14], Gaál, Pethő and Pohst added the corresponding index forms:

Case 1. $(m, n) \equiv (m_1, n_1) \equiv (1, 1) \pmod 4$,
integral basis: $\left\{1,\ (1+\sqrt{m})/2,\ (1+\sqrt{n})/2,\ \left(1+\sqrt{m}+\sqrt{n}+\sqrt{m_1 n_1}\right)/4\right\}$
discriminant: $D_K = (lm_1 n_1)^2$
index form:

$$I(x_2, x_3, x_4) = \left(l\left(x_2 + \frac{x_4}{2}\right)^2 - \frac{n_1}{4}x_4^2\right)\left(l\left(x_3 + \frac{x_4}{2}\right)^2 - \frac{m_1}{4}x_4^2\right) \times \left(n_1\left(x_3 + \frac{x_4}{2}\right)^2 - m_1\left(x_2 + \frac{x_4}{2}\right)^2\right)$$

Case 2. $(m,n)\equiv(1,1)\pmod 4$, $(m_1,n_1)\equiv(3,3)\pmod 4$
integral basis: $\left\{1,\ (1+\sqrt{m})/2,\ (1+\sqrt{n})/2,\ \left(1-\sqrt{m}+\sqrt{n}+\sqrt{m_1n_1}\right)/4\right\}$
discriminant: $D_K=(lm_1n_1)^2$
index form:

$$I(x_2,x_3,x_4)=\left(l\left(x_2-\frac{x_4}{2}\right)^2-\frac{n_1}{4}x_4^2\right)\left(l\left(x_3+\frac{x_4}{2}\right)^2-\frac{m_1}{4}x_4^2\right)$$
$$\times\left(n_1\left(x_3+\frac{x_4}{2}\right)^2-m_1\left(x_2-\frac{x_4}{2}\right)^2\right)$$

Case 3. $(m,n)\equiv(1,2)\pmod 4$
integral basis: $\left\{1,\ (1+\sqrt{m})/2,\ \sqrt{n},\ \left(\sqrt{n}+\sqrt{m_1n_1}\right)/2\right\}$
discriminant: $D_K=(4lm_1n_1)^2$
index form:

$$I(x_2,x_3,x_4)=\left(lx_2^2-n_1x_4^2\right)\left(l\left(x_3+\frac{x_4}{2}\right)^2-\frac{m_1}{4}x_4^2\right)$$
$$\times\left(4n_1\left(x_3+\frac{x_4}{2}\right)^2-m_1x_2^2\right)$$

Case 4. $(m,n)\equiv(2,3)\pmod 4$
integral basis: $\left\{1,\ \sqrt{m},\ \sqrt{n},\ \left(\sqrt{m}+\sqrt{m_1n_1}\right)/2\right\}$
discriminant: $D_K=(8lm_1n_1)^2$
index form:

$$I(x_2,x_3,x_4)=\left(\frac{l}{2}(2x_2+x_3)^2-\frac{n_1}{2}x_4^2\right)\left(2lx_3^2-\frac{m_1}{2}x_4^2\right)$$
$$\times\left(2n_1x_3^2-\frac{m_1}{2}(2x_2+x_4)^2\right)$$

Case 5. $(m,n)\equiv(3,3)\pmod 4$
integral basis: $\left\{1,\ \sqrt{m},\ (\sqrt{m}+\sqrt{n})/2,\ \left(1+\sqrt{m_1n_1}\right)/2\right\}$
discriminant: $D_K=(4lm_1n_1)^2$
index form:

$$I(x_2,x_3,x_4)=\left(l(2x_2+x_3)^2-n_1x_4^2\right)\left(lx_3^2-m_1x_4^2\right)$$
$$\times\left(\frac{n_1}{4}x_3^2-m_1\left(x_2+\frac{x_3}{2}\right)^2\right)$$

Finding the minimal index $\mu(K)$ is equivalent to determining the minimal $\mu\in\mathbb{N}$ for which the equation

$$I(x_2,x_3,x_4)=\pm\mu \quad \text{in} \quad x_2,x_3,x_4\in\mathbb{Z} \tag{4}$$

is solvable. For $x_2,x_3,x_4\in\mathbb{Z}$ the quadratic factors of the index form admit integral values. Fix the order of the factors in above index forms and denote the absolute value of the first, second and third factor by $F_1 = F_1(x_2, x_3, x_4)$, $F_2 = F_3(x_2, x_3, x_4)$, $F_3 = F_3(x_2, x_3, x_4)$, respectively. That means we want to find integers x_2, x_3, x_4 such that the product $F_1F_2F_3$ is minimal. It can be easily shown that F_1, F_2, F_3, according to cases $1-5$ are related in the following way (see [16, Lemma 1])

Lemma 2 *The following hold:*

$$\begin{aligned} \textit{Cases } 1,2,4: &\quad \pm F_1 m_1 \pm F_2 n_1 = \pm F_3 l \\ \textit{Case } 3: &\quad \pm F_1 m_1 \pm 4F_2 n_1 = \pm F_3 l \\ \textit{Case } 5: &\quad \pm F_1 m_1 \pm F_2 n_1 = \pm 4F_3 l \end{aligned}$$

By Lemma 2 among the three possible equations only two are independent. In the totally real case the index form is the product of tree factors F_1, F_2, F_3, of "Pellian type". In this case Gaál, Pethő and Pohst [16] gave following algorithm for finding the minimal index and all elements with minimal index. Consider system of equations obtained by equating the first quartic factor of the index form with $\pm F_1$ and second factor with $\pm F_2$. The system of these two equations can be written as

$$Ax^2 - By^2 = C \tag{5}$$

$$Dx^2 - Fz^2 = G \quad \text{in} \quad x, y, z \in \mathbb{Z}, \tag{6}$$

where the values of A, B, C, D, F, G and the new variables x, y, z are listed in the following table

Case	A	B	C	D	F	G	x	y	z
1	n_1	l	$\pm 4F_1$	m_1	l	$\pm 4F_2$	x_4	$2x_2 + x_4$	$2x_3 + x_4$
2	n_1	l	$\pm 4F_1$	m_1	l	$\pm 4F_2$	x_4	$2x_2 - x_4$	$2x_3 + x_4$
3	n_1	l	$\pm F_1$	m_1	l	$\pm 4F_2$	x_4	x_2	$2x_3 + x_4$
4	n_1	l	$\pm 2F_1$	$m_1/2$	$2l$	$\pm F_2$	x_4	$2x_2 + x_4$	x_3
5	n_1	l	$\pm F_1$	m_1	l	$\pm F_2$	x_4	$2x_2 + x_4$	x_3

Note that m_1 is even in Case 4. In each particular case, first we find the field index $m(K)$ which we can easy calculate from [14, Theorem 4]. We proceed with $\mu = \nu \cdot m(K)$ ($\nu = 1, 2, ...$). For each such μ we try to find positive integers F_1, F_2, F_3 with $\mu = F_1F_2F_3$ satisfying the corresponding relation of Lemma 2. If there exist such F_1, F_2, F_3, then we calculate all such triples. For each such triple we determine all solutions of the corresponding system (5) and (6). If none of these systems of equations have solutions, then we proceed to the next ν, otherwise μ is the minimal index and collecting all solutions of systems of equations corresponding to valid factors F_1, F_2, F_3 of μ we get all solutions of (4), i.e. we obtain all integral elements with minimal index in K.

3 Finding minimal index

Let $c \geq 3$ be positive integer such that c, $4c+1$, $c-1$ are square-free integers relatively prime in pairs. Let $m = m_1 l$, $n = n_1 l$ where $m_1, n_1, l \in \{c, 4c+1, c-1\}$

are distinct integers. Then field (1) is totally real bicyclic biquadratic field.

In order to prove Theorem 1 we will use a method of Gaál, Pethő and Pohst [16] given in previous section. Since they distinguished five cases according to the congruence behavior of m, n, m_1, n_1 modulo 4, we have to observe following cases:

i) If $c \equiv 0 \pmod 4$ or $c \equiv 1 \pmod 4$ then c or $c-1$ is not square free integer, respectively;

ii) If $c \equiv 2 \pmod 4$, $m_1 = 4c+1$, $n_1 = c$ and $l = c-1$, then $n_1 \equiv 2 \pmod 4$, $m_1 \equiv 1 \pmod 4$, $l \equiv 1 \pmod 4$ which implies $m \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 4$. Therefore, we obtain the system

$$(c-1)\,V^2 - cU^2 = \pm F_1, \tag{7}$$

$$(4c+1)\,V^2 - cZ^2 = \pm F_3, \tag{8}$$

$$(4c+1)\,U^2 - (c-1)\,Z^2 = \pm 4F_2, \tag{9}$$

where

$$U = x_4,\ V = x_2,\ Z = 2x_3 + x_4, \tag{10}$$

and from Lemma 2 we obtain that

$$\pm(4c+1)\,F_1 \pm (c-1)\,F_3 = \pm 4cF_2 \tag{11}$$

must hold. In this case the integral basis of K_c is

$$\left\{1,\ \frac{1+\sqrt{(4c+1)\,(c-1)}}{2},\ \sqrt{c\,(c-1)},\ \frac{\sqrt{c\,(c-1)}+\sqrt{(4c+1)\,c}}{2}\right\}$$

and its discriminant is $D = (4c\,(4c+1)\,(c-1))^2$.

iii) Let $c \equiv 3 \pmod 4$, $n_1 = 4c+1$, $m_1 = c-1$, $l = c$. Then $n_1 \equiv 1 \pmod 4$, $m_1 \equiv 2 \pmod 4$, $l \equiv 3 \pmod 4$ which implies $m = m_1 l \equiv 2 \pmod 4$ and $n = n_1 l \equiv 3 \pmod 4$. In this case, we have the system

$$(4c+1)\,U^2 - cV^2 = \pm 2F_1 \tag{12}$$

$$(c-1)\,U^2 - 4cZ^2 = \pm 2F_2 \tag{13}$$

$$4\,(4c+1)\,Z^2 - (c-1)\,V^2 = \pm 2F_3 \tag{14}$$

where

$$U = x_4,\ V = 2x_2 + x_4,\ Z = x_3, \tag{15}$$

and from Lemma 2 we obtain that

$$\pm(c-1)\,F_1 \pm (4c+1)\,F_2 = \pm cF_3 \tag{16}$$

must hold. The integral basis of K_c is

$$\left\{1,\ \sqrt{c\,(c-1)},\ \sqrt{c\,(4c+1)},\ \frac{\sqrt{c\,(c-1)}+\sqrt{(4c+1)\,(c-1)}}{2}\right\}$$

and its discriminant is $D=(8c\,(4c+1)\,(c-1))^2$.

Now we will calculate the field index $m\,(K_c)$ of K_c. First we form differences $d_1=m_1-l$, $d_2=n_1-l$, $d_3=m_1-n_1$. We have:

ii) $d_1=3c+2$, $d_2=1$, $d_3=3c+1$ if $c\equiv 2\,(\mathrm{mod}\,4)$,

iii) $d_1=-1$, $d_2=3c+1$, $d_3=-3c-2$ if $c\equiv 3\,(\mathrm{mod}\,4)$.

In both cases, we find neither 3 nor 4 divides all three differences d_1, d_2, d_3, therefrom, according [14, Theorem 4], we conclude $m\,(K_c)=1$. Therefore, we have proved statement *i)* of Theorem 1.

Note that if $c\leq 83$ then $c\in\{3,7,14,15,22,23,34,35,39,43,58,59,62,67,79,78\}$ since c, $4c+1$, $c-1$ are square free positive integers relatively prime in pairs. Therefore, according to this fact, we will suppose that $c\geq 14$ if $c\equiv 2\,(\mathrm{mod}\,4)$.

Now we will formulate our strategy of searching the minimal index $\mu\,(K_c)=:\mu\,(c)$ and all elements with minimal index. Finding of minimal index $\mu\,(c)$ is equivalent to finding system of above forms with minimal product $F_1F_2F_3$ which has solution. It is obvious that our fields are not monogenic since the necessary condition $m_1n_1\equiv(-1)^{\delta}\,(\mathrm{mod}\,4)$, $\delta=0,1$, is not satisfied (see [17]).

Observe that if $(\pm F_1,\pm 4F_2,\pm F_3)=(-4,20,4)$, then system (7), (8) and (9) has solutions $(U,V,Z)=(\pm 2,\pm 2,\pm 4)$ which implies that $\mu\,(c)\leq 80$ for all $c\equiv 2\,(\mathrm{mod}\,4)$.

Similarly, if $(\pm 2F_1,\pm 2F_2,\pm 2F_3)=(4,-4,20)$, then system (12), (13) and (14) has solutions $(U,V,Z)=(\pm 2,\pm 4,\pm 1)$ which implies that $\mu\,(c)\leq 40$ for all $c\equiv 3\,(\mathrm{mod}\,4)$.

Also, if $c=3$ and $(\pm 2F_1,\pm 2F_2,\pm 2F_3)=(10,2,-2)$, then system (12), (13) and (14) has solutions $(U,V,Z)=(\pm 1,\pm 1,0)$ which implies that $\mu\,(3)\leq 5$. In [16] it can be found that $\mu\,(3)=5$ and all elements with minimal index are given by $(x_2,x_3,x_4)=\pm(-1,0,1),\pm(0,0,1)$.

Therefore, it is natural to conjecture that for all $c\equiv 3\,(\mathrm{mod}\,4)$, c large enough, corresponding fields have the same minimal index, i.e. that minimal index doesn't depend of c if c is large enough. Similarly for $c\equiv 2\,(\mathrm{mod}\,4)$. Therefore, we will suppose that $F_1F_2F_3\leq 80$ if $c\equiv 2\,(\mathrm{mod}\,4)$, $c\geq 14$ and $F_1F_2F_3\leq 40$ if $c\equiv 3\,(\mathrm{mod}\,4)$, $c\geq 7$.

In both cases, first we use theory of continued fractions in order to determine all possible small values of the right hand side of the first two equations of our systems such that the system of these two equations has solutions. In particular, we will use a characterization in terms of continued fractions of α of all fractions a/b satisfying the inequality

$$\left|\alpha - \frac{a}{b}\right| < \frac{M}{b^2},$$

where $M \in \mathbb{N}$, $M \leq 5$. For all pairs $(\pm F_1, \pm F_3)$ or $(\pm F_1, \pm F_2)$ obtained in this way, using corresponding relations (11) or (16), respectively, we will calculate all possible triples $(\pm F_1, \pm F_2, \pm F_3)$ for which our systems may have solutions. Then for each obtained triple $(\pm F_1, \pm F_2, \pm F_3)$ we have to find are corresponding systems solvable or not. Of all solvable systems that are obtained, we choose system (or systems) with minimal product $F_1F_2F_3$. Then minimal index $\mu(c)$ is equal to that minimal product $F_1F_2F_3$ and solutions of that system (or these systems) leads to all integral elements with minimal index.

3.1 Case $c \equiv 3 \pmod 4$

Let $c \equiv 3 \pmod 4$, $c \geq 7$. First suppose that (U, V, Z) is nonnegative integer solution of the system of equations (12), (13) and (14) with $F_1F_2F_3 \leq 40$. Observe that if one of the integers U, V, Z is equal to zero, then (12), (13) and (14) imply that other two integers are not equal to zero.

i) If $U = 0$, then (12) and (13) imply

$$\begin{aligned} -cV^2 &= \pm 2F_1, \\ -4cZ^2 &= \pm 2F_2. \end{aligned}$$

Therefrom we have $F_1F_2 = c^2Z^2V^2 \leq 40$ and V is even. Since $c \geq 7$, $V^2 \geq 4$ and $Z \neq 0$ we obtain a contradiction.

ii) If $Z = 0$, then (12), (13) and (14) imply

$$\begin{aligned} (4c+1)\,U^2 - cV^2 &= \pm 2F_1, \\ (c-1)\,U^2 &= \pm 2F_2, \\ -(c-1)\,V^2 &= \pm 2F_3. \end{aligned}$$

Therefrom we have $F_2F_3 = \frac{(c-1)^2}{4}U^2V^2 \leq 40$. Since U, $V \neq 0$ we obtain a contradiction if $c \neq 7, 11$. If $c = 7$, then $F_2F_3 = 9U^2V^2 \leq 40$ which implies $(U, V) = (1,1)\,, (1,2)\,, (2,1)\,.$ If $c = 11$, then $F_2F_3 = 25U^2V^2 \leq 40$ which implies $(U, V) = (1,1)\,.$ Additionally, we have

$$F_1F_2F_3 = \left|\frac{1}{2}(4c+1)\,U^2 - \frac{1}{2}cV^2\right| \cdot \frac{(c-1)^2}{4} \cdot V^2U^2 \leq 40. \qquad (17)$$

Now, for $c = 7$ and $(U, V) = (1,1)\,, (1,2)\,, (2,1)$ inequality (17) implies a contradiction. Similarly, we obtain a contradiction for $c = 11$ and $(U, V) = (1,1)\,.$

iii) If $V=0$, then (12) and (14) imply

$$(4c+1)\,U^2 = \pm 2F_1$$
$$4\,(4c+1)\,Z^2 = \pm 2F_3.$$

Therefrom we have $F_1F_3 = (4c+1)^2\,U^2Z^2 \leq 40$ and U is even. Since $c \geq 7$, $U^2 \geq 4$ and $Z \neq 0$ we obtain a contradiction.

Let (U,V,Z) be positive integer solution of the system of Pellian equations

$$(4c+1)\,U^2 - cV^2 = \lambda_1, \tag{18}$$
$$(c-1)\,U^2 - 4cZ^2 = \lambda_2. \tag{19}$$

where λ_1 and λ_2 are non-zero integers such that $|\lambda_1| \leq 3M_1c$ and $|\lambda_2| \leq 3M_2\,(c-1)$, where $M_1,\ M_2 \in \mathbb{N}$, $M_1 \leq 4$, $M_2 \leq 5$. Then $\frac{V}{U}$ is a good rational approximation of $\sqrt{\frac{4c+1}{c}}$ and $\frac{U}{Z}$ is a good rational approximation of $\sqrt{\frac{4c}{c-1}}$. First of all, we have $\frac{V}{U} \geq 1$. Indeed, if $V < U$, then $(4c+1)(V+1)^2 - cV^2 \leq 3M_1c$ which is a contradiction. Similarly, $\frac{U}{Z} \geq 1$, since for $U < Z$ we obtain $4c\,(U+1)^2 - (c-1)\,U^2 \leq 3M_2\,(c-1)$ which implies a contradiction. Therefore, we find that

$$V + \sqrt{\frac{4c+1}{c}}U \geq U + U\sqrt{4+\frac{1}{c}} > U + 2U = 3U,$$

which implies

$$\left|\sqrt{\frac{4c+1}{c}} - \frac{V}{U}\right| = \left|\frac{4c+1}{c} - \frac{V^2}{U^2}\right| \cdot \left|\sqrt{\frac{4c+1}{c}} + \frac{V}{U}\right|^{-1}$$
$$< \frac{|\lambda_1|}{cU^2} \cdot \frac{1}{3} \leq \frac{M_1}{U^2}.$$

Similarly,

$$U + \sqrt{\frac{4c}{c-1}}Z \geq Z + Z\sqrt{4+\frac{4}{c-1}} > 3Z$$

implies

$$\left|\sqrt{\frac{4c}{c-1}} - \frac{U}{Z}\right| = \left|\frac{4c}{c-1} - \frac{U^2}{Z^2}\right| \cdot \left|\sqrt{\frac{4c}{c-1}} + \frac{U}{Z}\right|^{-1}$$
$$< \frac{|\lambda_2|}{(c-1)\,Z^2} \cdot \frac{1}{3} \leq \frac{M_2}{Z^2}.$$

Proposition 3 *Let $c \equiv 3 \pmod 4$, $c \geq 7$. Let (U,V,Z) be positive integer solution of the system of Pellian equations (12) and (13) where $\gcd(U,V) = d$, $\gcd(U,Z) = g$ and $F_1, F_2 \leq 40$. Then*

$$F_1 \leq \frac{3}{2}M_1cd^2 \quad \textit{and} \quad F_2 \leq \frac{3}{2}M_2\,(c-1)\,g^2,$$

where $M_1 = 4$ *and* $M_2 = 5$ *for* $c = 7$; $M_1 = M_2 = 3$ *for* $c = 11$; $M_1 = M_2 = 2$ *for* $c = 15, 19, 23$ *and* $M_1 = M_2 = 1$ *for* $c \geq 35$.

Proof. If $c \geq 35$, than we have

$$F_1 \leq 40 < \frac{3}{2} \cdot 1 \cdot 35 \cdot 1^2 \leq \frac{3}{2} M_1 c d^2$$

and

$$F_2 \leq 40 < \frac{3}{2} \cdot 1 \cdot (35 - 1) \cdot 1^2 \leq \frac{3}{2} M_2 \left(c - 1\right) g^2.$$

Similarly for the cases $c = 7, 11, 15, 19, 23$. ■

The simple continued fraction expansion of a quadratic irrational $\alpha = \frac{a+\sqrt{d}}{b}$ is periodic. This expansion can be obtained using the following algorithm. Multiplying the numerator and the denominator by b, if necessary, we may assume that $b|(d - a^2)$. Let $s_0 = a$, $t_0 = b$ and

$$a_n = \left\lfloor \frac{s_n + \sqrt{d}}{t_n} \right\rfloor, \quad s_{n+1} = a_n t_n - s_n, \quad t_{n+1} = \frac{d - s_{n+1}^2}{t_n} \quad \text{for } n \geq 0 \tag{20}$$

(see [21, Chapter 7.7]). If $(s_j, t_j) = (s_k, t_k)$ for $j < k$, then

$$\alpha = [a_0, \ldots, a_{j-1}, \overline{a_j, \ldots, a_{k-1}}].$$

Applying this algorithm to quadratic irrationals

$$\sqrt{\frac{4c+1}{c}} = \frac{\sqrt{c(4c+1)}}{c} \quad \text{and} \quad \sqrt{\frac{4c}{c-1}} = \frac{\sqrt{4c(c-1)}}{c-1}$$

we find that

$$\sqrt{\frac{4c+1}{c}} = \left[2, \overline{4c, 4}\right], \text{ where } (s_0, t_0) = (0, c),$$
$$(s_1, t_1) = (2c, 1), (s_2, t_2) = (2c, c), (s_3, t_3) = (2c, 1)$$

and

$$\sqrt{\frac{4c}{c-1}} = \left[2, \overline{c-1, 4}\right], \text{ where } (s_0, t_0) = (0, c-1),$$
$$(s_1, t_1) = (2(c-1), 4), (s_2, t_2) = (2(c-1), c-1), (s_3, t_3) = (2(c-1), 4).$$

Let p_n/q_n denote the nth convergent of α. The following result of Worley [23] and Dujella [5] extends classical results of Legendere and Fatou concerning Diophantine approximations of the form $\left|\alpha - \frac{a}{b}\right| < \frac{1}{2b^2}$ and $\left|\alpha - \frac{a}{b}\right| < \frac{1}{b^2}$.

Theorem 4 (Worley [23], Dujella [5]) *Let* α *be a real number and* a *and* b *coprime nonzero integers, satisfying the inequality*

$$\left|\alpha - \frac{a}{b}\right| < \frac{M}{b^2},$$

where M *is a positive real number. Then* $(a, b) = (rp_{n+1} \pm up_n, rq_{n+1} \pm uq_n)$, *for some* $n \geq -1$ *and nonnegative integers* r *and* u *such that* $ru < 2M$.

Explicit versions of Theorem 4 for $M=2$, was given by Worley [23, Corollary, p. 206]. Recently, Dujella and Ibrahimpašić [6, Propositions 2.1 and 2.2] extended Worley's work and gave explicit and sharp versions of Theorem 4 for $M=3,4,...,12$.

We would like to apply Theorem 4 in order to determine all values of λ_1 with $|\lambda_1| \leq 3M_1 c$, $M_1 \in \mathbb{N}$, $M_1 \leq 4$ for which equation (18) has solution and all values of λ_2 with $|\lambda_2| \leq 3M_2\,(c-1)$, $M_2 \in \mathbb{N}$, $M_2 \leq 5$ for which equation (19) has solutions. We need following lemma (see [8, Lemma 1])

Lemma 5 *Let $\alpha\beta$ be a positive integer which is not a perfect square, and let p_n/q_n denotes the nth convergent of continued fraction expansion of $\sqrt{\frac{\alpha}{\beta}}$. Let the sequences (s_n) and (t_n) be defined by (20) for the quadratic irrational $\frac{\sqrt{\alpha\beta}}{\beta}$. Then*

$$\alpha(rq_{n+1}+uq_n)^2-\beta(rp_{n+1}+up_n)^2=(-1)^n(u^2t_{n+1}+2rus_{n+2}-r^2t_{n+2}). \quad (21)$$

Since the period length of the continued fraction expansions of both $\sqrt{\frac{4c+1}{c}}$ and $\sqrt{\frac{4c}{c-1}}$ is equal to 2, according to Lemma 5, we have to consider only the fractions $(rp_{n+1}+up_n)/(rq_{n+1}+uq_n)$ for $n=0$ and $n=1$. By checking all possibilities, it is now easy to prove the following results.

Proposition 6 *Let $c \equiv 3 \,(\mathrm{mod}\, 4)$, $c \geq 7$ and λ_1 be an non-zero integer such that $|\lambda_1| \leq 3M_1 c$ and such that the equation (18) has a solution in relatively prime integers V and Z.*

i) If $c \geq 35$ and $M_1 = 1$ then

$$\lambda_1 \in A_1(c) = \{1, -c\}.$$

ii) If $c = 15, 19, 23$ and $M_1 = 2$ then

$$\lambda_1 \in A_1(c) = \{1, -c, 3c+1, 1-5c, 4c+1\}$$

iii) If $c = 11$ and $M_1 = 3$ then

$$\begin{aligned}\lambda_1 \in A_1(11) &= \{1, -c, 3c+1, 1-5c, 4c+1, 4+7c, 4-9c\}\\ &= \{1, -11, 34, -54, 45, 81, -95\}.\end{aligned}$$

iv) If $c = 7$ and $M_1 = 4$ then

$$\begin{aligned}\lambda_1 \in A_1(7) &= \{1, -c, 3c+1, 4c+1, 1-5c, 4+7c, 4-9c, 9-13c\}\\ &= \{1, -7, 22, -34, 29, 53, -59, -83, -82\}.\end{aligned}$$

Proposition 7 *Let $c \equiv 3 \,(\mathrm{mod}\, 4)$, $c \geq 7$ and λ_2 be an non-zero integer such that $|\lambda_2| \leq 3M_2\,(c-1)$ and such that the equation (19) has a solution in relatively prime integers U and Z.*

i) If $c \geq 35$ and $M_2 = 1$ then

$$\lambda_2 \in A_2(c) = \{-4, c-1\}.$$

ii) If $c = 15, 19, 23$ and $M_2 = 2$ then

$$\lambda_2 \in A_2(c) = \{-4, c-1, -3c-1, -4c, 5c-9\}$$

iii) If $c = 11$ and $M_2 = 3$ then

$$\begin{aligned} \lambda_2 \in A_2(11) &= \{-4, c-1, -3c-1, -4c, 5c-9, -7c-9, 9c-25\} \\ &= \{-4, 10, -34, -44, 46, -86, 74\}. \end{aligned}$$

iv) If $c = 7$ and $M_2 = 5$ then

$$\begin{aligned} \lambda_2 \in A_2(7) &= \left\{ \begin{array}{c} -4, c-1, -3c-1, -4c, 5c-9, -7c-9, 9c-25, \\ 12c-16, 13c-49, 17c-81, 21c-121 \end{array} \right\} \\ &= \{-4, 6, -22, -28, 26, -58, 38, 68, 42\} \end{aligned}$$

Corollary 8 *Let $c \equiv 3 \pmod 4$, $c \geq 7$.*

i) Let (U, V) be positive integer solution of the equation (12) such that $\gcd(U, V) = d$ and $F_1 \leq \frac{3}{2} M_1 c d^2$ where $M_1 = 4$ for $c = 7$, $M_1 = 3$ for $c = 11$, $M_1 = 2$ for $c = 15$, 19, 23 and $M_1 = 1$ for $c \geq 35$. Then

$$\pm 2F_1 \in \left\{ \lambda_1 d^2 : \lambda_1 \in A_1(c) \right\},$$

where sets $A_1(c)$ are given in Proposition 6.

ii) Let (U, Z) be positive integer solution of the equation (13) such that $\gcd(U, Z) = g$ and $F_2 \leq \frac{3}{2} M_2 (c-1) g^2$ where $M_2 = 5$ for $c = 7$, $M_2 = 3$ for $c = 11$, $M_2 = 2$ for $c = 15$, 19, 23 and $M_2 = 1$ for $c \geq 35$. Then

$$\pm 2F_2 \in \left\{ \lambda_2 g^2 : \lambda_2 \in A_2(c) \right\},$$

where sets $A_2(c)$ are given in Proposition 7.

Proof. Directly from Propositions 6 and 7. ∎

Proposition 9 *Let $c \equiv 3 \pmod 4$, $c \geq 7$. Let (U, V, Z) be positive integer solution of the system of Pellian equations (12) and (13) where $\gcd(U, V) = d$, $\gcd(U, Z) = g$ and $F_1, F_2 \leq 40$. Then*

i)

$$(\pm F_1, \pm F_2) \in B(c) \times D(c)$$

where $B(c) = B_0 \cup B_1(c)$, $D(c) = D_0 \cup D_1(c)$ and

$$B_0 = \{2, 8, 18, 32\}, \quad D_0 = \{-2, -8, -18, -32\},$$

$$B_1(c)=\emptyset \quad \textit{if} \quad c\geq 35, \quad B_1(c)=\left\{\frac{3c+1}{2}\right\}=\{35\} \quad \textit{if} \quad c=23,$$

$$B_1(c)=\left\{\frac{3c+1}{2},-2c\right\}=\{29,-38\} \quad \textit{if} \quad c=19,$$

$$B_1(c)=\left\{\frac{3c+1}{2},-2c,\frac{1-5c}{2}\right\} \quad \textit{if} \quad c\leq 15,$$

$$D_1(c)=\emptyset \quad \textit{if} \quad c\geq 83, \quad D_1(c)=\left\{\frac{c-1}{2}\right\} \quad \textit{if} \quad 35\leq c\leq 79,$$

$$D_1(c)=\left\{\frac{c-1}{2},-\frac{3c+1}{2}\right\}=\{-35,11\} \quad \textit{if} \quad c=23,$$

$$\begin{aligned} D_1(c)&=\left\{\frac{c-1}{2},-\frac{3c+1}{2},2(c-1),-2c\right\}\\ &=\{-38,-29,9,36\} \quad \textit{if} \quad c=19, \end{aligned}$$

$$\begin{aligned} D_1(c)&=\left\{\frac{c-1}{2},-\frac{3c+1}{2},2(c-1),-2c,\frac{5c-9}{2}\right\}=\\ &=\{-30,-23,7,28,33\} \quad \textit{if} \quad c=15, \end{aligned}$$

$$\begin{aligned} D_1(c)&=\left\{\frac{c-1}{2},-\frac{3c+1}{2},2(c-1),-2c,\frac{5c-9}{2},\frac{9c-25}{2}\right\}\\ &=\{-22,-17,5,20,23,37\} \quad \textit{if} \quad c=11, \end{aligned}$$

$$\begin{aligned} D_1(c)&=\left\{\begin{array}{c} \frac{c-1}{2},-\frac{3c+1}{2},2(c-1),-2c,\frac{5c-9}{2},\\ \frac{9}{2}(c-1),-\frac{7c+9}{2},\frac{9c-25}{2},\frac{13c-49}{2},6c-8 \end{array}\right\}=\\ &=\{-29,-14,-11,3,12,13,19,21,27,34\} \quad \textit{if} \quad c=7. \end{aligned}$$

ii) Additionally, if $F_1F_2\leq 40$, *then* $(\pm F_1,\pm F_2)\in S(c)$ *where* $S(c)=S_0\cup S_1(c)$ *and*

$$S_0=\{(2,-2),(2,-8),(2,-18),(8,-2),(18,-2)\}$$

$$S_1(c)=\emptyset \textit{ for } c\geq 83;$$

$$S_1(c)=\left\{\left(2,\frac{c-1}{2}\right)\right\} \textit{ for } 15\leq c\leq 79;$$

$$S_1(c)=\{(2,5),(2,-17),(2,20),(8,5),(17,-2)\} \textit{ for } c=11;$$

$$S_1(c)=\left\{\begin{array}{c} (2,3),(2,-11),(2,-14),(2,12),(2,13),(2,19),\\ (8,3),(11,-2),(11,3),(-14,-2),(-17,-2) \end{array}\right\} \textit{ for } c=7;$$

Proof.

i) From Proposition 3 and Corollary 8 we have $\pm 2F_1 \in \left\{\lambda_1 d^2 : \lambda_1 \in A_1(c)\right\}$ and $\pm 2F_2 \in \left\{\lambda_2 g^2 : \lambda_2 \in A_2(c)\right\}$ where sets $A_1(c)$ and $A_2(c)$ are given in Propositions 6.and 7, respectively. Therefore,

a) For all $c \geq 7$ we have $\pm 2F_1 = d^2, -cd^2$. Additionally, we have $\pm 2F_1 = (3c+1)\,d^2,\ (1-5c)\,d^2,\ (4c+1)\,d^2$ if $c \leq 23$, $\pm 2F_1 = (4+7c)\,d^2$, $(4-9c)\,d^2$if $c = 11, 7$ and $\pm 2F_1 = (1-12c)\,d^2,\ (9-13c)\,d^2$, i.e. $\pm 2F_1 = -83d^2, -82d^2$ if $c = 7$. Since $F_1 \leq 40$, we obtain:

- **i.** $F_1 = \frac{d^2}{2} \leq 40$ implies $d = 2, 4, 6, 8$, i.e. $\pm F_1 = 2, 8, 18, 32$;
- **ii.** $F_1 = \frac{cd^2}{2} \leq 40$ implies $d \leq \frac{4\sqrt{5}}{\sqrt{c}} \leq \frac{4\sqrt{5}}{\sqrt{7}} < 4$ and d is even, i.e. $d = 2$. Thus, $\pm F_1 = -2c$ for $c \leq 19$ since $F_1 = 2c > 40$ if $c \geq 23$;
- **iii.** $F_1 = \frac{(3c+1)d^2}{2} \leq 40$ implies $d \leq 4\frac{\sqrt{5}}{\sqrt{3c+1}} \leq 4\frac{\sqrt{5}}{\sqrt{3\cdot 7+1}} < 2$, i.e. $d = 1$. Thus, $\pm F_1 = \frac{3c+1}{2}$ for $c \leq 23$;
- **iv.** $F_1 = \frac{(5c-1)d^2}{2} \leq 40$ implies $d \leq 4\frac{\sqrt{5}}{\sqrt{5c-1}} \leq 4\frac{\sqrt{5}}{\sqrt{5\cdot 7-1}} < 2$, i.e. $d = 1$. Thus, $\pm F_1 = \frac{1-5c}{2}$ for $c \leq 15$;
- **v.** $F_1 = \frac{(4c+1)d^2}{2} \leq 40$ implies $d \leq 4\frac{\sqrt{5}}{\sqrt{4c+1}} \leq 4\frac{\sqrt{5}}{\sqrt{3\cdot 7+1}} < 2$ and d is even, i.e. there is no solution;
- **vi.** $F_1 = \frac{(7c+4)d^2}{2} \leq 40$ implies $d \leq 4\frac{\sqrt{5}}{\sqrt{7c+4}} \leq 4\frac{\sqrt{5}}{\sqrt{7\cdot 7+4}} < 2$ and d is even, i.e. there is no solution;
- **vii.** $F_1 = \frac{(9c-4)d^2}{2} \leq 40$ implies $d \leq 4\frac{\sqrt{5}}{\sqrt{9c-4}} \leq 4\frac{\sqrt{5}}{\sqrt{9\cdot 7-4}} < 2$ and d is even, i.e. there is no solution;
- **viii.** $F_1 = \frac{83d^2}{2} \leq 40$ implies $d < 1$, i.e. there is no solution.
- **ix.** $F_1 = \frac{82d^2}{2} \leq 40$ implies $d < 1$, i.e. there is no solution.

Therefrom, we obtain sets $B(c)$.

b) For all $c \geq 7$ we have $\pm 2F_2 = -4g^2,\ (c-1)\,g^2$. Additionally, we have $\pm 2F_2 = -(3c+1)\,g^2,\ -4cg^2,\ (5c-9)\,g^2$ if $c \leq 23$, $\pm 2F_2 = (-7c-9)\,g^2, (9c-25)\,g^2$ if $c \leq 11$ and $\pm 2F_1 = (12c-16)\,g^2,\ (13c-49)\,g^2$, i.e. $\pm 2F_1 = 68g^2, 42g^2$ if $c = 7$. Since $F_2 \leq 40$, we obtain

- **i.** $F_2 = 2g^2 \leq 40$ implies $g = 1, 2, 3, 4$, i.e. $\pm F_2 = -2, -8, -18, -32$;
- **ii.** $F_2 = \frac{c-1}{2}g^2 \leq 40$ implies $g \leq 4\frac{\sqrt{5}}{\sqrt{c-1}} \leq \frac{4\sqrt{5}}{\sqrt{7}} < 4$, i.e. $g = 1, 2, 3$. Thus, we have $\pm F_2 = \frac{c-1}{2}$ if $c \leq 79$, $\pm F_2 = 2\,(c-1)$ if $c \leq 19$ and $\pm F_2 = \frac{9}{2}\,(c-1) = 27$ if $c = 7$;
- **iii.** $F_2 = \frac{3c+1}{2}g^2 \leq 40$ implies $g \leq 4\frac{\sqrt{5}}{\sqrt{3c+1}} \leq 4\frac{\sqrt{5}}{\sqrt{3\cdot 7+1}} < 2$, i.e. $g = 1$. Thus, we have $\pm F_2 = -\frac{3c+1}{2}$ for $c \leq 23$;
- **iv.** $F_2 = 2cg^2 \leq 40$ implies $g \leq \frac{2\sqrt{5}}{\sqrt{c}} \leq \frac{2\sqrt{5}}{\sqrt{7}} < 2$, i.e. $g = 1$. Thus, we have $\pm F_2 = -2c$ if $c \leq 19$;
- **v.** $F_2 = \frac{5c-9}{2}g^2 \leq 40$ implies $g \leq 4\frac{\sqrt{5}}{\sqrt{5c-9}} \leq 4\frac{\sqrt{5}}{\sqrt{5\cdot 7-9}} < 2$, i.e. $g = 1$. Thus, we have $\pm F_2 = \frac{5c-9}{2}$ if $c \leq 15$;

vi. $F_2 = \frac{7c+9}{2} g^2 \leq 40$ implies $g \leq 4\frac{\sqrt{5}}{\sqrt{7c+9}} \leq 4\frac{\sqrt{5}}{\sqrt{7 \cdot 7+9}} < 2$, i.e. $g = 1$. Thus, we have $\pm F_2 = -\frac{7c+9}{2}$ if $c = 7$;

vii. $F_2 = \frac{9c-25}{2} g^2 \leq 40$ implies $g \leq 4\frac{\sqrt{5}}{\sqrt{9c-25}} \leq 4\frac{\sqrt{5}}{\sqrt{9 \cdot 7-25}} < 2$, i.e. $g = 1$. Thus, we have $\pm F_2 = \frac{9c-25}{2}$ if $c \leq 11$;

viii. $F_2 = (6c-8)\, g^2 = 34g^2 \leq 40$ implies $g = 1$. Thus, we have $\pm F_2 = 34$ if $c = 7$;

ix. $F_2 = \frac{13c-49}{2} g^2 = 21g^2 \leq 40$ implies $g = 1$. Thus, we have $\pm F_2 = 21$ if $c = 7$;

Therefrom, we get sets $D(c)$.

ii) Directly from *i)* since $S(c) = \{(s,t) \in B(c) \times D(c) : |s| \cdot |t| \leq 40\}$.

■

If system (12), (13) and (14) has solution for some positive integers F_1, F_2, F_3, $F_1 F_2 F_3 \leq 40$, then $(\pm F_1, \pm F_2) \in S(c)$, where set $S(c)$ is given in Proposition 9 and triple $(\pm F_1, \pm F_2, \pm F_3)$ satisfies one of the equations in (16). First, for each pair $(\pm F_1, \pm F_2) \in S(c)$ we check if there exist $F_3 \in \mathbb{N}$, $F_1 F_2 F_3 \leq 40$ such that any of the equations (16) holds. For all pairs of the form $(\pm F_1, \pm F_2) = (s,t)$ condition $F_1 F_2 F_3 \leq 40$ is satisfied if $F_3 \in F(s,t) = \left\{k \in \mathbb{N} : k \leq \frac{40}{|s||t|}\right\}$. Therefore, for each pair $(s,t) \in S(c)$ and for each $k \in F(s,t)$, we have to check if any of these four equations

$$s(c-1) + t(4c+1) = \pm kc \quad \text{or} \quad s(c-1) - t(4c+1) = \pm kc \tag{22}$$

holds. For example, $(\pm F_1, \pm F_2) = (2,-2) \in S(c)$ for all $c \geq 7$. From (22) we obtain

$$-6c - 4 = \pm F_3 c \quad \text{or} \quad 10c = \pm F_3 c.$$

Since $F_3 \in F(2,-2) = \{k \in \mathbb{N} : k \leq 10\}$ the only possibility is $\pm F_3 = 10$. Furthermore, $(\pm F_1, \pm F_2) = (2,-8) \in S(c)$ for all $c \geq 7$. From (22) we obtain

$$-30c - 10 = \pm F_3 c \quad \text{or} \quad 34c + 6 = \pm F_3 c,$$

which implies a contradiction if $F_3 \in F(2,-8) = \{1,2\}$. We proceed similarly for all elements of $S(c)$. The only triple we obtain on this way is $(\pm F_1, \pm F_2, \pm F_3) = (2,-2,10)$ and the corresponding system is

$$(4c+1)\, U^2 - cV^2 = 4 \tag{23}$$

$$(c-1)\, U^2 - 4cZ^2 = -4 \tag{24}$$

$$4(4c+1)\, Z^2 - (c-1)\, V^2 = 20. \tag{25}$$

Since this system has solution $(U,V,Z) = (\pm 2, \pm 4, \pm 1)$, we have $\mu(c) = 40$ for all $c \equiv 3 \pmod 4$, $c \geq 7$.

Next step is finding all elements with minimal index. Therefore we have to solve the above system.

If $c = 3$ and $(\pm 2F_1, \pm 2F_2, \pm 2F_3) = (10, 2, -2)$, then system (12), (13) and (14) has solutions $(U, V, Z) = (\pm 1, \pm 1, 0)$ which implies that $\mu(3) \leq 5$. Therefore, if we suppose that $F_1 F_2 F_3 \leq 5$ and use the same procedure as for the case $c \geq 7$, we obtain that the only possibility is $(\pm F_1, \pm F_2, \pm F_3) = (5, 1, -1)$ and the corresponding system is

$$13U^2 - 3V^2 = 10 \tag{26}$$

$$U^2 - 6Z^2 = 1 \tag{27}$$

$$V^2 - 26Z^2 = 1. \tag{28}$$

This system has solutions $(U, V, Z) = (\pm 1, \pm 1, 0)$ which implies that $\mu(3) = 5$. In [1] Anglin showed that system (27) and (28) has only the trivial solutions $(U, V, Z) = (\pm 1, \pm 1, 0)$. Now using (15), we find that all integral elements with minimal index are given by $(x_2, x_3, x_4) = \pm(-1, 0, 1), \pm(0, 0, 1)$.

3.2 Case $c \equiv 2 \pmod 4$

For all $c \equiv 2 \pmod 4$, $c \geq 14$, in similar way, we obtain that the only solvable system of the form (7), (8) and (9) with $F_1 F_2 F_3 \leq 80$ is

$$(c-1)V^2 - cU^2 = -4, \tag{29}$$

$$(4c+1)V^2 - cZ^2 = 4, \tag{30}$$

$$(4c+1)U^2 - (c-1)Z^2 = 20. \tag{31}$$

Since this system has solution $(U, V, Z) = (\pm 2, \pm 2, \pm 4)$, we have $\mu(c) = 80$ for all $c \equiv 2 \pmod 4$, $c \geq 14$. In order to find all elements with minimal index, we have to find all solutions to the system (29), (30) and (31).

Therefore, we have proved statement ii) of Theorem 1.

4 Finding all elements with minimal index

Now, we have to solve systems that are obtained in Section 3. These systems are very suitable for application of method given in [7]. We will prove following result

Theorem 10 *i) Let* $c \equiv 2 \pmod 4$, $c \geq 14$ *be an integer. The only solutions to system (29), (30) and (31) are* $(U, V, Z) = (\pm 2, \pm 2, \pm 4)$.

ii) Let $c \equiv 3 \pmod 4$, $c \geq 7$ *be an integer. The only solutions to system (23), (24) and (25) are* $(U, V, Z) = (\pm 2, \pm 4, \pm 1)$.

Therefrom we have following corollary which finishes the proof of Theorem 1.

Corollary 11 *Let $c \geq 7$ be positive integer such that c, $4c+1$, $c-1$ are square-free integers relatively prime in pairs. Then all integral elements with minimal index in the field $K_c = \mathbb{Q}\left(\sqrt{(4c+1)\,c}, \sqrt{c\,(c-1)}\right)$ are given by:*

i) $(x_2, x_3, x_4) = \pm(1, \pm 1, 2), \pm(3, \pm 1, -2)$ *if* $c \equiv 3 \,(\mathrm{mod}\, 4)$;

ii) $(x_2, x_3, x_4) = \pm(\pm 2, 1, 2),\ \pm(\pm 2, 3, -2)$ *if* $c \equiv 2 \,(\mathrm{mod}\, 4)$;

Proof.

i) Let $c \equiv 3 \,(\mathrm{mod}\, 4)$, $c \geq 7$. Since all solutions of the system (23), (24) and (25) are given by $(U, V, Z) = (\pm 2, \pm 4, \pm 1)$ and since in this case we have $U = x_4, V = 2x_2 + x_4, Z = x_3$, we obtain

$$x_4 = \pm 2,\ 2x_2 + x_4 = \pm 4,\ x_3 = \pm 1,$$

which implies $(x_2, x_3, x_4) = (1, \pm 1, 2), (3, \pm 1, -2), (-1, \pm 1, -2), (-3, \pm 1, 2)$.

ii) Let $c \equiv 2 \,(\mathrm{mod}\, 4)$. Since all solutions of the system (29), (30) and (31) are given by $(U, V, Z) = (\pm 2, \pm 2, \pm 4)$ and since in this case we have $U = x_4, V = x_2, Z = 2x_3 + x_4$, we obtain

$$x_4 = \pm 2,\ x_2 = \pm 2,\ 2x_3 + x_4 = \pm 4,$$

which implies $(x_2, x_3, x_4) = (\pm 2, 1, 2), (\pm 2, -1, -2), (\pm 2, 3, -2), (\pm 2, -3, 2)$.

■

Observe that if $c \equiv 2 \,(\mathrm{mod}\, 4)$ and (U, V, Z) is a solution to system (29), (30) and (31), then all integers U, V, Z are even. Let $Z = 2Z_1$. Then system (29) and (30) is equivalent to the system

$$(c-1)\,V^2 - cU^2 = -4, \tag{32}$$

$$(4c+1)\,V^2 - 4cZ_1^2 = 4. \tag{33}$$

If $c \equiv 3 \,(\mathrm{mod}\, 4)$ and (U, V, Z) is a solution to system (23), (24) and (25), then integers U, V are even. Let $Z_2 = 2Z$, $V = 2V_2$. Then system (23) and (24) is equivalent to the system

$$(c-1)\,U^2 - cZ_2^2 = -4, \tag{34}$$

$$(4c+1)\,U^2 - 4cV_2^2 = 4. \tag{35}$$

Therefore, in order to prove Theorem 10 it is enough to analyze system (2) and (3), where $c \geq 7$ and prove the following theorem:

Theorem 12 *Let $c \geq 2$ be an integer. Then the only solutions to system of Pellian equations (2) and (3) are $(X, Y, T) = (\pm 2, \pm 2, \pm 2)$.*

In order to prove Theorem 12, first we will find a lower bound for solutions of this system using the "congruence method" introduced in [9]. The comparison of this lower bound with an upper bound obtained from a theorem of Bennett [4] on simultaneous approximations of algebraic numbers finishes the proof for $c \geq 89037$. For $c \leq 89036$ we use a theorem a Baker and Wüstholz [3] and a version of the reduction procedure due to Baker and Davenport [2].

Lemma 13 *Let (X, Y, T) be positive integer solution of the system of Pellian equations (2) and (3). Then there exist nonnegative integers m and n such that*

$$X = v_m = w_n,$$

where the sequences (v_m) and (w_n) are given by

$$v_0 = 2, \quad v_1 = 8c - 2, , \quad v_{m+2} = 2\,(2c-1)\, v_{m+1} - v_m, \quad m \geq 0 \tag{36}$$

and

$$w_0 = 2, \quad w_1 = 32c + 2, \quad w_{n+2} = 2\,(8c+1)\, w_{n+1} - w_n, \quad n \geq 0. \tag{37}$$

Proof. Let $Y_1 = cY$, $T_1 = 2cT$, then system (2) and (3) is equivalent to system

$$Y_1^2 - c\,(c-1)\,X^2 = 4c, \tag{38}$$
$$T_1^2 - c\,(4c+1)\,X^2 = -4c. \tag{39}$$

Since $\sqrt{c\,(4c+1)} = \left[2c, \overline{4, 4c}\right]$ and $\sqrt{c\,(c-1)} = \left[c-1, \overline{2, 2\,(c-1)}\right]$ we find that fundamental solutions of equations

$$A^2 - c\,(c-1)\,B^2 = 1, \tag{40}$$
$$D^2 - c\,(4c+1)\,B^2 = 1, \tag{41}$$

are given by $a_1 + b_1\sqrt{c\,(c-1)} = 2c - 1 + 2\sqrt{c\,(c-1)}$, $a_2 + b_2\sqrt{c\,(4c+1)} = 8c + 1 + 4\sqrt{c\,(4c+1)}$, respectively. By [19, Theorem 108], it follows that if $u_0 + v_0\sqrt{c\,(c-1)}$ is the fundamental solution of the class $\mathcal{C}$ of equation (38), than inequalities

$$0 < |u_0| \leq \sqrt{\frac{1}{2}\,(a_1+1)\cdot 4c} = \sqrt{\frac{1}{2}\,(2c-1+1)\,4c} = 2c,$$
$$0 \leq v_0 \leq \frac{b_1}{\sqrt{2\,(a_1+1)}}\sqrt{4c} = \frac{2}{\sqrt{2\,(2c-1+1)}}\sqrt{4c} = 2$$

must hold. This implies that

$$u_0 + v_0\sqrt{c\,(c-1)} = 2c + 2\sqrt{c\,(c-1)} \quad \text{and} \quad u_0' + v_0'\sqrt{c\,(c-1)} = -2c + 2\sqrt{c\,(c-1)}$$

are possible fundamental solution of equation (38). Since

$$u_0 u_0' \equiv c\,(c-1)\,v_0 v_0' \pmod{4c} \quad \text{and} \quad u_0 v_0' \equiv u_0' v_0 \pmod{4c},$$

these solutions belong to the same class (see [19, Chapter VI, 58.]), so we have only one fundamental solution $u_0 + v_0\sqrt{c\,(c-1)} = 2c + 2\sqrt{c\,(c-1)}$. Similarly, by [19, Theorem 108a], we find that if $s_0 + w_0\sqrt{c\,(4c+1)}$ is the fundamental solution of the class $\mathcal{C}$ of equation (39), than inequalities

$$0 \le |s_0| \le \sqrt{\frac{1}{2}\,(a_1-1)\cdot 4c} = \sqrt{\frac{1}{2}\,(8c+1-1)\,4c} = 4c,$$
$$0 < w_0 \le \frac{b_2}{\sqrt{2\,(a_1-1)}}\sqrt{4c} = \frac{4}{\sqrt{2\,(8c+1-1)}}\sqrt{4c} = 2.$$

must hold. This implies that

$$s_0+w_0\sqrt{c\,(4c+1)} = 4c+2\sqrt{c\,(4c+1)} \text{ and } s_0+w_0\sqrt{c\,(4c+1)} = -4c+2\sqrt{c\,(4c+1)}$$

are possible fundamental solution of equation (38). Since

$$s_0 s_0' \equiv c\,(4c+1)\,w_0 w_0' (\operatorname{mod} 4c) \quad \text{and} \quad s_0 w_0' \equiv s_0' w_0 (\operatorname{mod} 4c),$$

these solutions belong to the same class, therefore we have only one fundamental solution $s_0 + w_0\sqrt{c\,(4c+1)} = 4c + 2\sqrt{c\,(c-1)}$. Now, the theory of Pellian equations guarantees that all solutions of (2) and (3) are given by (36) and (37), respectively. ■

Therefore, in order to prove Theorem 12, it suffices to show that $v_m = w_n$ implies $m = n = 0$.

Solving recurrences (36) and (37) we find

$$v_m = \frac{1}{\sqrt{c-1}}\Big[(\sqrt{c}+\sqrt{c-1})\Big(2c-1+2\sqrt{c(c-1)}\Big)^m - (\sqrt{c}-\sqrt{c-1})\Big(2c-1-2\sqrt{c(c-1)}\Big)^m\Big], \tag{42}$$
$$w_n = \frac{1}{\sqrt{4c+1}}\Big[(2\sqrt{c}+\sqrt{4c+1})\Big(8c+1+4\sqrt{c(4c+1)}\Big)^n - (2\sqrt{c}-\sqrt{4c+1})\Big(8c+1-4\sqrt{c(4c+1)}\Big)^n\Big]. \tag{43}$$

4.1 Congruence relations

Now, we will find a lower bound for nontrivial solutions using the "congruence method".

Lemma 14 *Let the sequences (v_m) and (w_n) be defined by (36) and (37). Then for all $m, n \ge 0$ we have*

$$v_m \equiv (-1)^{m-1}\,(4m(m+1)c-2) \pmod{32c^2}, \tag{44}$$
$$w_n \equiv 16n(n+1)c+2 \pmod{512c^2}. \tag{45}$$

Proof. Both relations are obviously true for $m, n \in \{0, 1\}$.
Assume that (44) is valid for $m-2$ and $m-1$. Then

$$\begin{aligned} v_m =& 2\,(2c-1)\,v_{m-1} - v_{m-2} \\ \equiv& 2\,(2c-1)\,[(-1)^{m-2}\,(4\,(m-1)\,mc-2)] - [(-1)^{m-3}\,(4(m-2)(m-1)c-2)] \\ \equiv& (-1)^{m-1}\,[-16c^2m\,(m-1) + 4cm\,(m+1) - 2] \\ =& (-1)^{m-1}\,(4m(m+1)c-2) \pmod{32c^2}. \end{aligned}$$

Assume that (45) is valid for $n-2$ and $n-1$. Then

$$\begin{aligned} w_n =&\ 2\,(8c+1)\,w_{n-1} - w_{n-2} \\ \equiv& 2\,(8c+1)\,[16n(n-1)c+2] - [16\,(n-1)\,(n-2)c+2] \\ \equiv& 16cn^2 + 16cn + 2 \\ =& [16n(n+1)c+2] \pmod{512c^2}. \end{aligned}$$

■

Suppose that m and n are positive integers such that $v_m = w_n$. Then, of course, $v_m \equiv w_n \,(\mathrm{mod}\ 32c^2)$. By Lemma 14, we have $(-1)^m \equiv 1 \pmod{2c}$ and therefore m is even.

Assume that $m(m+1) < \frac{8}{5}c$. Since $n \leq m$ we also have $n(n+1) < \frac{8}{5}c$. Furthermore, Lemma 14 implies

$$-4m(m+1)c + 2 \equiv 16n(n+1)c + 2 \pmod{32c^2}$$

and

$$-\frac{m(m+1)}{2} \equiv 2n(n+1) \pmod{4c}. \tag{46}$$

Consider the positive integer

$$A = 2n(n+1) + \frac{m(m+1)}{2}.$$

We have $0 < A < 4c$ and, by (46), $A \equiv 0 \,(\mathrm{mod}\ 2c)$, a contradiction.

Hence $m(m+1) \geq \frac{8}{5}c$ and it implies $m > \sqrt{1.6c} - 0.5$. Therefore we proved

Proposition 15 *If $v_m = w_n$ and $n \neq 0$, then $m > \sqrt{1.6c} - 0.5$.*

4.2 An application of a theorem of Bennett

It is clear that the solutions of the system (38) and (39) induce good rational approximations to the numbers

$$\theta_1 = \sqrt{\frac{c-1}{c}} \quad \text{and} \quad \theta_2 = \sqrt{\frac{4c+1}{4c}}\,.$$

More precisely, we have

Lemma 16 *All positive integer solutions* (X, Y, T) *of the system of Pellian equations (38) and (39) satisfy*

$$|\theta_1 - \frac{Y}{X}| < \frac{4}{\sqrt{c\,(c-1)}} \cdot X^{-2}\,,$$
$$|\theta_2 - \frac{T}{X}| < \frac{1}{c} \cdot X^{-2}.$$

Proof. We have

$$\left|\sqrt{\frac{c-1}{c}}X + Y\right| \geq \frac{1}{2}\left(\left|\sqrt{\frac{c-1}{c}}X + Y\right| + \left|\sqrt{\frac{c-1}{c}}X - Y\right|\right) \geq \frac{1}{2}\left|2\sqrt{\frac{c-1}{c}}X\right|$$
$$= X\sqrt{\frac{c-1}{c}},$$

which implies

$$\left|\sqrt{\frac{c-1}{c}} - \frac{Y}{X}\right| = \left|\frac{c-1}{c} - \frac{Y^2}{X^2}\right| \cdot \left|\sqrt{\frac{c-1}{c}} + \frac{Y}{X}\right|^{-1}$$
$$< \frac{4}{cX^2} \cdot \sqrt{\frac{c}{c-1}} = \frac{4}{\sqrt{c\,(c-1)}} \cdot X^{-2}$$

Similarly,

$$\left|T + \sqrt{\frac{4c+1}{4c}}X\right| \geq X\sqrt{1 + \frac{1}{4c}} \geq X,$$

implies

$$\left|\sqrt{\frac{4c+1}{4c}} - \frac{T}{X}\right| = \left|\frac{4c+1}{4c} - \frac{T^2}{X^2}\right| \cdot \left|\sqrt{\frac{4c+1}{4c}} + \frac{T}{X}\right|^{-1} < \frac{4}{4cX^2} \leq \frac{1}{c} \cdot X^{-2}.$$

■

The numbers θ_1 and θ_2 are square roots of rationals which are very close to 1. For simultaneous Diophantine approximations to such kind of numbers there are very useful effective results of Masser and Rickert [18] and Bennett [4]. We will use the following theorem of Bennett [4, Theorem 3.2].

Theorem 17 *If* a_i, p_i, q *and* N *are integers for* $0 \leq i \leq 2$, *with* $a_0 < a_1 < a_2$, $a_j = 0$ *for some* $0 \leq j \leq 2$, q *nonzero and* $N > M^9$, *where*

$$M = \max_{0 \leq i \leq 2}\{|a_i|\} \geq 3,$$

then we have

$$\max_{0 \leq i \leq 2}\left\{\left|\sqrt{1 + \frac{a_i}{N}} - \frac{p_i}{q}\right|\right\} > (130N\gamma)^{-1}q^{-\lambda}$$

where

$$\lambda = 1 + \frac{\log(32.04N\gamma)}{\log\left(1.68N^2 \prod_{0\le i<j\le 2}(a_i - a_j)^{-2}\right)}$$

and

$$\gamma = \begin{cases} \frac{(a_2-a_0)^2(a_2-a_1)^2}{2a_2-a_0-a_1} & \text{if } a_2 - a_1 \ge a_1 - a_0, \\ \frac{(a_2-a_0)^2(a_1-a_0)^2}{a_1+a_2-2a_0} & \text{if } a_2 - a_1 < a_1 - a_0. \end{cases}$$

We will apply Theorem 17 with $a_0 = -4$, $a_1 = 0$, $a_2 = 1$, $N = 4c$, $M = 4$, $q = X$, $p_0 = Y$, $p_1 = X$, $p_2 = T$. If $c \ge 65\,537$, then the condition $N > M^9$ is satisfied and we obtain

$$(130 \cdot 4c \cdot \frac{400}{9})^{-1} X^{-\lambda} < \frac{4}{\sqrt{c\,(c-1)}} \cdot X^{-2}\,. \tag{47}$$

If $c \ge 84762$ then $2 - \lambda > 0$ and (47) implies

$$\log X < \frac{11.782}{2-\lambda}\,. \tag{48}$$

Furthermore,

$$\frac{1}{2-\lambda} = \frac{1}{1 - \frac{\log(5696c)}{\log(0.0672\,c^2)}} < \frac{\log\left(0.0672c^2\right)}{\log(0.000011797c)}\,.$$

On the other hand, from (43) we find that

$$w_n > \left(2c - 1 + 2\sqrt{c(c-1)}\right)^m > (4c-3)^m,$$

and Proposition 15 implies that if $(m,n) \ne (0,0)$, then

$$X > (4c-3)^{\sqrt{1.6c}-0.5}\,.$$

Therefore,

$$\log X > (\sqrt{1.6c} - 0.5)\log(4c-3). \tag{49}$$

Combining (48) and (49) we obtain

$$\sqrt{1.6c} - 0.5 < \frac{11.782\log\left(0.0672\,c^2\right)}{\log(4c-3)\log(0.000011797c)} \tag{50}$$

and (50) yields to a contradiction if $c \ge 89037$. Therefore we proved

Proposition 18 *If c is an integer such that $c \ge 89037$, then the only solution of the equation $v_m = w_n$ is $(m,n) = (0,0)$.*

4.3 The Baker-Davenport method

In this section we will apply so called Baker-Davenport reduction method in order to prove Theorem 12 for $2 \le c \le 89036$.

Lemma 19 *If $v_m = w_n$ and $n \neq 0$, then*

$$0 < m \log\left(2c-1+2\sqrt{c(c-1)}\right) - n \log\left(8c+1+4\sqrt{c(4c+1)}\right)$$
$$+ \log \frac{\sqrt{4c+1}(\sqrt{c}+\sqrt{c-1})}{\sqrt{c-1}(2\sqrt{c}+\sqrt{4c+1})} < 0.313\,74\,\left(8c+1+4\sqrt{c(4c+1)}\right)^{-2n}.$$

Proof. In standard way (for e.g. see [7, Lemma 5]) ■

Now we will apply the following theorem of Baker and Wüstholz [3]:

Theorem 20 *For a linear form $\Lambda \neq 0$ in logarithms of l algebraic numbers $\alpha_1, \ldots, \alpha_l$ with rational integer coefficients $b_1, \ldots, b_l$ we have*

$$\log \Lambda \ge -18(l+1)!\, l^{l+1} (32d)^{l+2} h'(\alpha_1) \cdots h'(\alpha_l) \log(2ld) \log B\,,$$

where $B = \max\{|b_1|, \ldots, |b_l|\}$, and where d is the degree of the number field generated by $\alpha_1, \ldots, \alpha_l$.

Here

$$h'(\alpha) = \frac{1}{d} \max\{h(\alpha), |\log \alpha|, 1\}\,,$$

and $h(\alpha)$ denotes the standard logarithmic Weil height of α.

We will apply Theorem 20 to the form from Lemma 19. We have $l = 3$, $d = 4$, $B = m$,

$$\alpha_1 = 2c-1+2\sqrt{c(c-1)}, \qquad \alpha_2 = 8c+1+4\sqrt{c(4c+1)},$$

$$\alpha_3 = \frac{\sqrt{4c+1}(\sqrt{c}+\sqrt{c-1})}{\sqrt{c-1}(2\sqrt{c}+\sqrt{4c+1})}.$$

Under the assumption that $2 \le c \le 89036$ we find that

$$h'(\alpha_1) = \frac{1}{2}\log\alpha_1 < \frac{1}{2}\log 4c, \qquad h'(\alpha_2) = \frac{1}{2}\log\alpha_2 < 7.0848.$$

Furthermore, $\alpha_3 < 1.242\,7$, and the conjugates of α_3 satisfy

$$|\alpha_3'| = \frac{\sqrt{4c+1}(\sqrt{c}-\sqrt{c-1})}{\sqrt{c-1}(2\sqrt{c}+\sqrt{4c+1})} < 1,$$

$$|\alpha_3''| = \frac{\sqrt{4c+1}(-\sqrt{c}+\sqrt{c-1})}{\sqrt{c-1}(2\sqrt{c}-\sqrt{4c+1})} < 7.242\,7$$

$$|\alpha_3'''| = \frac{\sqrt{4c+1}(\sqrt{c-1}+\sqrt{c})(\sqrt{4c+1}+2\sqrt{c})}{\sqrt{c-1}} < 1424583.1.$$

Therefore,

$$h'(\alpha_3) < \frac{1}{4}\log\Big[(c-1)^2 \cdot 1.2427 \cdot 7.2427 \cdot 1424583.1\Big] < 9.7901.$$

Finally,

$$\log\left[0.31374\left(8c+1+4\sqrt{c(4c+1)}\right)^{-2n}\right] < -2n\log(4c)\,.$$

Hence, Theorem 20 implies

$$2n\log(4c) < 3.822\cdot 10^{15}\cdot\frac{1}{2}\cdot\log(4c)\cdot 7.0848.\cdot 9.7901\log m$$

and

$$\frac{n}{\log m} < 6.6275\cdot 10^{16}. \tag{51}$$

By Lemma 19, we have

$$\begin{aligned} m\log\left(2c-1+2\sqrt{c(c-1)}\right) &< n\log(8c+1+4\sqrt{c(4c+1)}) + 2.7188\times 10^{-4} \\ &< n\log\left[(8c+1+4\sqrt{c(4c+1)})\cdot 1.0004\right] \end{aligned}$$

and

$$\frac{m}{n} < 2.0003 \tag{52}$$

Combining (51) and (52), we obtain

$$\frac{m}{\log m} < \frac{2.0003\cdot n}{\log n} < 2.0003\cdot 6.6275\cdot 10^{16} < 1.3258\times 10^{17}$$

which implies $m < 5.7264\times 10^{18}$.

We may reduce this large upper bound using a variant of the Baker-Davenport reduction procedure [2]. The following lemma is a slight modification of [9, Lemma 5 a)]:

Lemma 21 *Assume that M is a positive integer. Let p/q be a convergent of the continued fraction expansion of κ such that $q > 10M$ and let $\varepsilon = \|\mu q\| - M\cdot\|\kappa q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality*

$$0 < m - n\kappa + \mu < AB^{-n}$$

in integers m and n with

$$\frac{\log(Aq/\varepsilon)}{\log B} \le n \le M\,.$$

We apply Lemma 21 with

$$\kappa = \frac{\log \alpha_2}{\log \alpha_1}, \qquad \mu = \frac{\log \alpha_3}{\log \alpha_1}, \qquad A = \frac{0.313\,74}{\log \alpha_1},$$

$$B = \left(8c + 1 + 4\sqrt{c(4c+1)}\right)^2 \quad \text{and} \quad M = 5.7264 \times 10^{18}.$$

If the first convergent such that $q > 10M$ does not satisfy the condition $\varepsilon > 0$, then we use the next convergent.

We performed the reduction from Lemma 21 for $2 \leq c \leq 89036$. The use of the second convergent was necessary in 3249 cases ($\approx 3.65\%$), the third convergent was used in 63 cases ($\approx 0.08\%$), the forth in 14 cases, the fifth in 2 cases and seventh convergent is used in only one case: $c = 69953$. In all cases we obtained $n \leq 6$. More precisely, we obtained $n \leq 6$ for $c \geq 2$; $n \leq 5$ for $c \geq 4$; $n \leq 4$ for $c \geq 6$; $n \leq 3$ for $c \geq 24$; $n \leq 2$ for $c \geq 256$; $n \leq 1$ for $c \geq 74211$. The next step of the reduction in all cases gives $n \leq 1$, which completes the proof.

Therefore, we proved

Proposition 22 *If c is an integer such that $2 \leq c \leq 89036$, then the only solution of the equation $v_m = w_n$ is $(m, n) = (0, 0)$.*

Proof of Theorem 12. The statement follows directly from Propositions 18 and 22.

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